Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL MONETARY POLICY IN PRODUCTION NETWORKS"

(*Econometrica*, Vol. 90, No. 3, May 2022, 1295–1336)

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THIS SUPPLEMENT contains two parts. Appendix B presents the proofs and derivations omitted from the main body of the paper. Appendix C contains robustness checks for the quantitative analysis in Section 5 of the paper.

APPENDIX B: PROOFS AND DERIVATIONS

This Appendix contains the proofs and derivations omitted from the main body of the paper. Throughout, with some abuse of notation, we write $\operatorname{diag}(x)$ to denote a diagonal matrix whose entries are equal to vector x, while we use $\operatorname{diag}(\mathbf{X})$ to denote a (column) vector whose elements are equal to the diagonal elements of matrix $\mathbf{X}$.

Proof of Proposition 2

We start by establishing equation (16). As a first observation, note that the optimality conditions of the representative household's problem are given by

$$V'\big(L(s)\big) = \mu(s)w(s), \tag{B.1}$$

$$U'\big(C(s)\big)\frac{\partial \mathcal{C}}{\partial c_i}(s) = \mu(s)p_i(\omega_i) \quad \text{for all } i, \tag{B.2}$$

where $\mu(s)$ is the Lagrange multiplier corresponding to the household's budget constraint. As a result,

$$V'\big(L(s)\big) = \frac{w(s)}{p_i(\omega_i)}U'\big(C(s)\big)\frac{\partial \mathcal{C}}{\partial c_i}(s). \tag{B.3}$$

Multiplying both sides of the above equation by $p_i(\omega_i)c_i(s)$ and summing over all i, we obtain

$$w(s) = m(s)\frac{V'\big(L(s)\big)}{C(s)U'\big(C(s)\big)}, \tag{B.4}$$

where we are using the assumption that the consumption aggregator $\mathcal{C}(\cdot)$ is a homogenous function of degree 1 and the fact that $\sum_{i=1}^n p_i(\omega_i)c_i(s) = m(s)$. Next, note that cost

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 https://doi.org/10.3982/ECTA18627

minimization by firms in industry i implies that[1]

$$\mathrm{mc}_i(s) = w(s)\big(z_i \cdot \partial F_i/\partial l_i(s)\big)^{-1}. \tag{B.5}$$

Replacing $w(s)$ from (B.4) into the above equation establishes (16).

We next show that if a feasible allocation is implementable as a sticky-price equilibrium, it must satisfy (13). Consider the price-setting problem of firm k in industry i, which sets its nominal price to maximize the expected value of its real profits:

$$\max_{p_{ik}(\omega_{ik})} \quad \mathbb{E}_{ik}\left[\frac{U'\big(C(s)\big)}{P(s)}\big((1-\tau_i)p_{ik}(\omega_{ik})y_{ik}(s) - \mathrm{mc}_i(s)y_{ik}(s)\big)\right], \tag{B.6}$$

$$\text{s.t.} \quad y_{ik}(s) = \big(p_{ik}(\omega_{ik})/p_i(\omega_i)\big)^{-\theta_i} y_i(s). \tag{B.7}$$

In the above problem, $U'(C(s))$ is the representative household's marginal utility of consumption and $P(s)$ is the nominal price of the consumption good bundle. Consequently, the nominal price set by the firm satisfies the following first-order condition:

$$\mathbb{E}_{ik}\left[U'\big(C(s)\big)\frac{\partial \mathcal{C}}{\partial c_i}(s)y_i(s)\left(\frac{p_{ik}(\omega_{ik})}{p_i(\omega_i)}\right)^{1-\theta_i}\left((1-\tau_i)\left(\frac{\theta_i-1}{\theta_i}\right) - \frac{\mathrm{mc}_i(s)}{p_{ik}(\omega_{ik})}\right)\right] = 0,$$

where we are using the fact that $\partial\mathcal{C}/\partial c_i(s) = p_i(\omega_i)/P(s)$.[2] Using equation (B.7) to express $p_{ik}(\omega_{ik})/p_i(\omega_i)$ in terms of quantities then implies that

$$p_{ik}(\omega_{ik}) = \left[(1-\tau_i)\left(\frac{\theta_i-1}{\theta_i}\right)\right]^{-1}\frac{\mathbb{E}_{ik}\big[v_{ik}(s)\,\mathrm{mc}_i(s)\big]}{\mathbb{E}_{ik}\big[v_{ik}(s)\big]}, \tag{B.8}$$

where $v_{ik}(s)$ is given by (17). Consequently, the nominal price set by the firm satisfies

$$p_{ik}(\omega_{ik}) = \frac{1}{\chi_i^s \varepsilon_{ik}(s)}\mathrm{mc}_i(s), \tag{B.9}$$

where $\chi_i^s = (1-\tau_i)(\theta_i - 1)/\theta_i$ is a wedge that arises due to government taxes/subsidies and monopolistic markups, whereas $\varepsilon_{ik}(s)$ is given by (15) and represents a wedge that arises due to the presence of nominal rigidities. Combining (B.9) with (B.3) and (B.7), we obtain

$$V'\big(L(s)\big) = \chi_i^s \varepsilon_{ik}(s) U'\big(C(s)\big)\frac{\partial\mathcal{C}}{\partial c_i}(s)\frac{w(s)}{\mathrm{mc}_i(s)}\left(\frac{y_{ik}(s)}{y_i(s)}\right)^{-1/\theta_i}.$$

Replacing for $\mathrm{mc}_i(s)$ from (B.5) into the above equation then establishes (13).

[1]The assumption that labor is an essential input in the production technology of all goods guarantees that $\partial F_i/\partial l_i(s) > 0$ for all i. Furthermore, note that the realized marginal cost of all firms in the same industry are identical, that is, $\mathrm{mc}_{ik}(s) = \mathrm{mc}_i(s)$ for all $s \in S$ and all firms k in industry i. As a result, $\partial F_i/\partial l_{ik}(s)$ is the same for all firms k in industry i. We therefore drop the firm index k in (B.5).

[2]This is a consequence of the household's optimization problem. Specifically, multiplying both sides of (B.2) by $c_i(s)$ and summing over all i implies that $\mu(s) = U'(C(s))/P(s)$, where we are using the assumption that $\mathcal{C}(\cdot)$ is a homogenous function of degree 1. Plugging this back into (B.2) then implies that $\partial\mathcal{C}/\partial c_i(s) = p_i(\omega_i)/P(s)$.

Finally, to establish (14), recall that the representative household's first-order optimality condition requires that (B.2) is satisfied for all i. As a result,

$$\frac{\partial \mathcal{C}}{\partial c_j}(s) = \frac{p_j(\omega_j)}{p_i(\omega_i)} \frac{\partial \mathcal{C}}{\partial c_i}(s) = \chi_i^s \varepsilon_{ik}(s) \frac{\partial \mathcal{C}}{\partial c_i}(s) \frac{p_j(\omega_j)}{\mathrm{mc}_i(s)} \left(\frac{y_{ik}(s)}{y_i(s)} \right)^{-1/\theta_i},$$

for all pairs of industries i and j, where once again we are using (B.7) and (B.9). Furthermore, whenever industry j is an input supplier of industry i, cost minimization by firm k in industry i implies that

$$\mathrm{mc}_i(s) = p_j(\omega_j)\big(z_i \cdot \partial F_i / \partial x_{ij,k}(s)\big)^{-1}.$$

The juxtaposition of the last two equations establishes (14).

Proof of Proposition 1

We establish this result as a special case of Proposition 2. Recall from Definitions 1 and 2 that a flexible-price equilibrium can be cast as a sticky-price equilibrium with an information structure under which the state $s = (z, \omega)$ is measurable with respect to the information set of all firms. As a result, in any flexible-price equilibrium, the right-hand side of (16) is measurable with respect to the information set of firm k in industry i, which in turn implies that $\mathbb{E}_{ik}[\mathrm{mc}_i(s)] = \mathrm{mc}_i(s)$. Consequently, (15) implies that $\varepsilon_{ik}(s) = 1$ for all $k \in [0, 1]$, all $i \in I$, and all $s \in S$. Plugging this into (13) and (14) and using the fact that all firms in the same industry set identical prices—and hence have identical outputs—then establishes (10) and (11).

Proof of Theorem 1

We prove the necessity claim as the converse claim is straightforward. Suppose there exists a flexible-price allocation indexed by $(\chi_1^f, \dots, \chi_n^f)$ that is also implementable as a sticky-price equilibrium. By Propositions 1 and 2, such an allocation simultaneously satisfies equations (10)–(11) and (13)–(14). As a result,

$$\chi_i^f = \chi_i^s \varepsilon_{ik}(s) \left(\frac{y_{ik}(s)}{y_i(s)} \right)^{-1/\theta_i}$$

for all states $s \in S$ and all firms $k \in [0, 1]$ in all industries i, where the wedge $\varepsilon_{ik}(s)$ is given by (15). In any flexible-price allocation, the outputs of all firms in the same industry are identical, i.e., $y_{ik}(s) = y_i(s)$ for all $k \in [0, 1]$. Therefore,

$$\chi_i^f = \chi_i^s \varepsilon_{ik}(s)$$

for all $s \in S$ and all firms k in all industries i. Note that, by assumption, scalars χ_i^f and χ_i^s, which are determined by fiscal policy, are invariant to the state s and do not depend on the firm index k. Therefore, the above equation holds only if $\varepsilon_{ik}(s)$ is also independent of s and k, that is, $\varepsilon_{ik}(s) = \varepsilon_i$ for all i.

Next, note that (15) guarantees that $\mathbb{E}_{ik}[v_{ik}(s)(\varepsilon_{ik}(s) - 1)] = 0$, where $v_{ik}(s)$ is given by (17). Consequently,

$$(\varepsilon_i - 1)\mathbb{E}_{ik}\big[v_{ik}(s)\big] = 0$$

for all i. But since $v_{ik}(s) > 0$ for all $s \in S$ in any feasible allocation, it must be the case that $\varepsilon_i = 1$ for all i. As a result, $\chi_i^s = \chi_i^f$ for all i. Furthermore, recall from the proof of Proposition 2 that, in any sticky-price equilibrium, each firm's price and marginal cost are related to one another via (B.9). Therefore,

$$p_{ik}(\omega_{ik}) = \frac{1}{\chi_i^f}\mathrm{mc}_i(s). \tag{B.10}$$

Equation (B.10) has three implications. First, given that its right-hand side is independent of k, it implies that all firms within the same industry set the same nominal price. Thus, we can write $p_{ik}(\omega_{ik}) = p_i(\omega_i)$, with the understanding that $p_i(\omega_i)$ is measurable with respect to the information set of all firms k in industry i. Second, (B.10) also implies that the marginal cost of industry i is measurable with respect to the information set of all firms in that industry. Finally, it establishes that, whenever an allocation can be implemented as both a sticky- and a flexible-price equilibrium, all firms employ constant markups to set their nominal prices. Consequently, we can write i's nominal price as a function of industry i's nominal input prices as

$$p_i(\omega_i) = \frac{1}{\chi_i^f z_i}K_i\big(w(s), p_1(\omega_1), \dots, p_n(\omega_n)\big), \tag{B.11}$$

where $K_i(\cdot)$ is a homogenous function of degree 1 and represents the cost function of firms in industry i. Dividing both sides of the above equation by the nominal wage leads to

$$p_i(\omega_i)/w(s) = \frac{1}{\chi_i^f z_i}K_i\big(1, p_1(\omega_1)/w(s), \dots, p_n(\omega_n)/w(s)\big). \tag{B.12}$$

We thus obtain a system of n equations and n unknowns that relates all industries' nominal prices relative to the nominal wage to productivity shocks $(z_1, \dots, z_n)$ and fiscal policy wedges $(\chi_1^f, \dots, \chi_n^f)$. Since, by assumption, labor is an essential input for the production technology of all industries, Theorem 1 of Stiglitz (1970) guarantees that there is at most one collection of relative prices that solves the system of equations in (B.12). In particular, for any industry i, there exists a unique function $h_i : \mathbb{R}_+^n \to \mathbb{R}_+$ such that

$$p_i(\omega_i) = w(s)h_i\big(\chi_1^f z_1, \dots, \chi_n^f z_n\big), \tag{B.13}$$

where $(h_1, \dots, h_n)$ solves the following system of equations:

$$h_i(z) = \frac{1}{z_i}K_i\big(1, h_1(z), \dots, h_n(z)\big) \quad \text{for all } i. \tag{B.14}$$

As we already established, the left-hand side of the (B.13) is measurable with respect to the information set of all firms in industry i. Therefore, a feasible allocation is implementable as an equilibrium under both flexible and sticky prices only if there exists a nominal function $w(s)$ such that

$$w(s)h_i\big(\chi_1^f z_1, \dots, \chi_n^f z_n\big) \in \sigma(\omega_{ik})$$

simultaneously for all firms k in all industries i.

The proof is therefore complete once we show that $h_i(z) = 1/g_i(z)$, where $g_i(z)$ is the marginal product of labor in the production of good i (as a function of realized productivity shocks) under the first-best allocation. To establish this, we make two observations. First, note that the system of equations in (B.14) is identical to marginal-cost pricing conditions in the economy's competitive equilibrium. Since, as we already discussed, this system of equations has at most one nonzero solution, h_i has to be equal to the marginal cost of good i relative to the wage in the economy's competitive equilibrium. Second, cost minimization implies that i's marginal product of labor is equal to the wage divided by the marginal cost of i. Putting these two observations together implies that h_i is equal to the reciprocal of i's marginal product of labor in the economy's competitive equilibrium. Finally, the fact that the economy's competitive equilibrium coincides with the first-best allocation (by the first welfare theorem) guarantees that $h_i(z) = 1/g_i(z)$, where $g_i(z)$ is the marginal product of labor as a function of productivity shocks in the first-best allocation.

Proof of Corollary 3

Suppose all firms in all industries $j \neq i$ set their prices under complete information about the state, s. As a result, $\mathbb{E}_{jk}[v_{jk}(s)] = v_{jk}(s)$ and $\mathbb{E}_{jk}[v_{jk}(s)\mathrm{mc}_j(s)] = v_{jk}(s)\mathrm{mc}_j(s)$ for all $k \in [0, 1]$ and all $j \neq i$. Therefore, (15) implies that

$$\varepsilon_{jk}(s) = 1 \quad \text{for all } j \neq i. \tag{B.15}$$

Let the monetary policy function $m(s)$ be given by

$$m(s) = M z_i \frac{U'\big(C(s)\big)C\big((s)\big)}{V'\big(L(s)\big)} \frac{\partial F_i}{\partial l_i}(s), \tag{B.16}$$

for some constant $M > 0$ that does not depend on the state, s. By (16), such a policy induces $\mathrm{mc}_i(s) = M$ for all s. Thus, (15) guarantees that $\varepsilon_{ik}(s) = 1$ for all firms $k \in [0, 1]$. This, together with (B.15), implies that the policy in (B.16) eliminates all wedges that are due to nominal rigidities, thus reducing equations (13)–(14) to (10)–(11). In other words, any flexible-price-implementable allocation can be implemented as part of a sticky-price equilibrium.

The proof is therefore complete once we show that the policy in (B.16) stabilizes the price of industry i. As we already established, such a policy induces $\mathrm{mc}_i(s) = M$ for all s. Thus, by equation (B.9), $p_{ik}(\omega_{ik}) = M/\chi_i^s$, which means that the nominal price set by firms in industry i is invariant to the economy's aggregate state.

Proof of Corollary 4

First, note that an aggregate labor-augmenting shock can be incorporated to our model in Section 2 by introducing an extra industry, labeled industry 0, that transforms household's labor supply into labor services sold to other industries. A TFP shock to this industry is identical to an aggregate labor-augmenting shock.

Next, recall from the proof of Theorem 1 that the marginal product of labor of industry i as a function of productivity shocks $(z_0, z_1, \dots, z_n)$ in the first-best allocation is given by $g_i(z) = 1/h_i(z)$, where $(h_0, h_1, \dots, h_n)$ satisfies (B.14) for all $i \in \{0, 1, \dots, n\}$. As a result,

$$h_i(z) = \frac{1}{z_i} K_i\big(1/z_0, h_1(z), \dots, h_n(z)\big) \quad \text{for all } i \in \{1, \dots, n\}.$$

Since the marginal cost function $K_i(\cdot)$ is homogenous of degree 1 for all i, it is immediate that the unique solution to the above system of equations is given by $h_i(z) = \frac{1}{z_0}\hat{h}_i(z_1, \dots, z_n)$, where $\hat{h}_i(\cdot)$ does not depend on z_0. Consequently, the marginal product of labor of industry i in the first-best allocation is equal to $g_i(z) = z_0/\hat{h}_i(z_1, \dots, z_n)$. Setting the nominal wage function $w(s) = z_0\hat{w}(z_1, \dots, z_n)$ for a function $\hat{w}(z_1, \dots, z_n)$ that does not depend on z_0 then implies that the left-hand side of (18) is

$$w(s)/g_i(\chi_1^f z_1, \dots, \chi_n^f z_n) = \hat{w}(z_1, \dots, z_n)\hat{h}_i(\chi_1^f z_1, \dots, \chi_n^f z_n),$$

which is, by assumption, measurable with respect to the information sets of all firms in industry i. Thus, by Theorem 1, any flexible-price allocation is implementable as a sticky-price equilibrium.

Proof of Proposition 3

Suppose the flexible-price allocation indexed by $(\chi_1^f, \dots, \chi_n^f)$ is implementable as a sticky-price equilibrium using a price-stabilization policy that assigns weight ψ_i to the price of industry i, that is,

$$\sum_{i=1}^n \psi_i \log p_i(\omega_i) + \left(1 - \sum_{i=1}^n \psi_i\right) \log w(s) = 0. \tag{B.17}$$

In the proof of Theorem 1, we established that a flexible-price allocation is implementable as a sticky-price equilibrium only if condition (B.11) is satisfied for all industries i. Under the assumption that the production technology of firms in industry i is given by (19), this condition is equivalent to $\log p_i(\omega_i) = -\log \chi_i^f - \log z_i + \alpha_i \log w(s) + \sum_{j=1}^n a_{ij} \log p_j(\omega_j)$. Solving for log nominal prices, we obtain

$$\log p_i(\omega_i) = \log w(s) - \sum_{j=1}^n \ell_{ij}\left(\log z_j + \log \chi_j^f\right), \tag{B.18}$$

where ℓ_{ij} denotes the (i, j) element of the Leontief inverse, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. Multiplying both sides of the above equation by ψ_i, summing over all i, and using (B.17), we obtain

$$\log w(s) = \sum_{i=1}^n \sum_{j=1}^n \psi_i \ell_{ij}\left(\log z_j + \log \chi_j^f\right).$$

Replacing the above into (B.18) implies that

$$\log p_{ik}(\omega_{ik}) = \sum_{r=1}^n \sum_{j=1}^n \psi_r \ell_{rj}\left(\log z_j + \log \chi_j^f\right) - \sum_{j=1}^n \ell_{ij}\left(\log z_j + \log \chi_j^f\right) \tag{B.19}$$

for all firms $k \in [0, 1]$ in industry i, where we are also using the fact that since the allocation is both flexible- and sticky-price implementable, all firms within the same industry set the same nominal price, that is, $p_{ik}(\omega_{ik}) = p_i(\omega_i)$. Since $\log p_{ik}(\omega_{ik})$ is measurable with respect to the information set of firm k in industry i, taking conditional expectations from both sides of (B.19) and subtracting the resulting equation from (B.19) implies that

$$\sum_{r=1}^{n}\sum_{j=1}^{n}\psi_r\ell_{rj}\big(\mathbb{E}_{ik}[\log z_j]-\log z_j\big)-\sum_{j=1}^{n}\ell_{ij}\big(\mathbb{E}_{ik}[\log z_j]-\log z_j\big)=0.$$

Rewrite the above equation as $(\psi'-u_i')\mathbf{L}(\mathbb{E}_{ik}[\log z]-\log z)=0$, where $\psi=(\psi_1,\dots,\psi_n)'$ and u_i denotes the ith unit vector. Multiplying both sides of this equation by $\log z'$ from the right and taking expectations with respect to the information set of firm k in industry i establishes (21).

Proof of Lemma 2

By (B.8), firm k in industry i sets a nominal price equal to $p_{ik}=\mathbb{E}_{ik}[\mathrm{mc}_i\, v_{ik}]/\mathbb{E}_{ik}[v_{ik}]$, where v_{ik} is given by (17) and we are using the assumption that $\tau_i=1/(1-\theta_i)$. Consequently,

$$\begin{aligned}\log p_{ik}-\mathbb{E}_{ik}[\log\mathrm{mc}_i]&=\log\mathbb{E}_{ik}\big[e^{\log v_{ik}-\mathbb{E}_{ik}[\log v_{ik}]+\log\mathrm{mc}_i-\mathbb{E}_{ik}[\log\mathrm{mc}_i]}\big]-\log\mathbb{E}_{ik}\big[e^{\log v_{ik}-\mathbb{E}_{ik}[\log v_{ik}]}\big]\\&=\frac{1}{2}\mathrm{var}_{ik}(\log\mathrm{mc}_i)+\mathrm{cov}_{ik}(\log\mathrm{mc}_i,\log v_{ik})+o(\delta^2)\end{aligned}$$

as $\delta\to 0$. Since the standard deviations of log productivity shocks in (22) and noise shocks in (23) scale linearly in δ, it follows that $\mathrm{var}_{ik}(\log\mathrm{mc}_i)=o(\delta)$ and $\mathrm{cov}_{ik}(\log\mathrm{mc}_i,\log v_{ik})=o(\delta)$, thus establishing (26).

To establish (27), recall that the production function of firms in industry i is given by (19), which implies that $\log\mathrm{mc}_i=\alpha_i\log w-\log z_i+\sum_{j=1}^{n}a_{ij}\log p_j$. This, together with (26), establishes (27).

Proof of Proposition 4

Since noise shocks $\epsilon_{ij,k}$ in firms' private signals in (23) are idiosyncratic and of order δ, the log-linearization (as $\delta\to 0$) of any industry-level or aggregate variable only depends on the productivity shocks. We thus let

$$\log w=\sum_{j=1}^{n}\kappa_j\log z_j+o(\delta), \tag{B.20}$$

$$\log p_i=\sum_{j=1}^{n}b_{ij}\log z_j+o(\delta) \tag{B.21}$$

denote, respectively, the log-linearization of the nominal wage and the nominal price of sectoral good i as $\delta\to 0$, where vector $\kappa=(\kappa_1,\dots,\kappa_n)'$ and matrix $\mathbf{B}=[b_{ij}]$ are to be determined. Furthermore, recall from Lemma 2 that, to a first-order approximation as $\delta\to 0$, the nominal price set by firm k in industry i is given by (27). Therefore,

$$\log p_{ik}=\alpha_i\sum_{j=1}^{n}\kappa_j\mathbb{E}_{ik}[\log z_j]-\mathbb{E}_{ik}[\log z_i]+\sum_{j=1}^{n}\sum_{r=1}^{n}a_{ij}b_{jr}\mathbb{E}_{ik}[\log z_r]+o(\delta)$$

$$= \alpha_i \sum_{j=1}^{n} \kappa_j \phi_{ik} \omega_{ij,k} - \phi_{ik} \omega_{ii,k} + \sum_{j=1}^{n} \sum_{r=1}^{n} a_{ij} b_{jr} \phi_{ik} \omega_{ir,k} + o(\delta), \tag{B.22}$$

where ϕ_{ik} is the degree of price flexibility of firm k in industry i given by (25). Integrating both sides of the above equation over all firms k in industry i implies that

$$\log p_i = \phi_i \alpha_i \sum_{j=1}^{n} \kappa_j \log z_j - \phi_i \log z_i + \phi_i \sum_{j=1}^{n} \sum_{r=1}^{n} a_{ij} b_{jr} \log z_r + o(\delta),$$

where $\phi_i = \int_0^1 \phi_{ik} \, \mathrm{d}k$ is the degree of price flexibility of industry i and we are using the fact that $\log p_i = \int_0^1 \log p_{ik} \, \mathrm{d}k + o(\delta)$. The juxtaposition of the above equation with equation (B.21) therefore implies that $\mathbf{B} = \mathbf{\Phi} \alpha \kappa' - \mathbf{\Phi} + \mathbf{\Phi} \mathbf{A} \mathbf{B}$, where $\mathbf{\Phi} = \mathrm{diag}(\phi)$. As a result,

$$\mathbf{B} = (\mathbf{I} - \mathbf{\Phi} \mathbf{A})^{-1} \mathbf{\Phi} \big(\alpha \kappa' - \mathbf{I} \big) = \mathbf{\Phi} (\mathbf{I} - \mathbf{A} \mathbf{\Phi})^{-1} \big(\alpha \kappa' - \mathbf{I} \big). \tag{B.23}$$

Multiplying both sides by $\log z$ and using equations (B.20) and (B.21) then establishes that industry-level prices satisfy (30). To establish (29), note that the vector of (log) nominal marginal costs is equal to $\log \mathrm{mc} = \alpha \log w - \log z + \mathbf{A} \log p$. Therefore, by (30), $\log \mathrm{mc} = (\mathbf{I} + \mathbf{A} \mathbf{\Phi} (\mathbf{I} - \mathbf{A} \mathbf{\Phi})^{-1})(\alpha \log w - \log z)$, which reduces to (29).

Proof of Proposition 6

We prove this result in three steps. First, we solve for household welfare in terms of nominal prices and the nominal wage. We then compare the result to welfare under the first-best allocation to obtain an expression for welfare loss, taking nominal prices as given. Finally, we provide a quadratic log-approximation to the welfare loss in terms of the cross-sectional mean and variance of firm-level pricing errors in (34) and (35). Given the indeterminacy of prices in the flexible-price equilibrium, we express the pricing errors under the normalization that the nominal wage is the same in the sticky- and flexible-price equilibria.

Expressing Welfare in Terms of Nominal Prices. As our first step, we obtain an expression for welfare as a function of all nominal prices and the nominal wage.

Recall from equation (B.7) that the output of firm k in industry i is given by $y_{ik} = y_i (p_{ik}/p_i)^{-\theta_i}$, whereas cost minimization implies that the firm's demand for the good produced by industry j is equal to $x_{ij,k} = a_{ij} y_{ik} \, \mathrm{mc}_i / p_j$. Therefore, total demand for the good produced by industry j by firms in industry i is $\int_0^1 x_{ij,k} \, \mathrm{d}k = a_{ij} p_i y_i \varepsilon_i / p_j$, where ε_i is a sectoral wedge and is given by [3]

$$\varepsilon_i = \frac{\mathrm{mc}_i}{p_i} \int_0^1 (p_{ik}/p_i)^{-\theta_i} \, \mathrm{d}k. \tag{B.24}$$

[3]This sectoral wedge is closely related to firm-level wedges ε_{ik} in Proposition 2. Specifically, under our assumption that taxes eliminate all steady-state distortions due to monopolistic markups, equation (B.9) implies that $\varepsilon_{ik} = \mathrm{mc}_i / p_{ik}$. Thus,

$$\log \varepsilon_i = \log \int_0^1 \varepsilon_{ik}^{\theta_i} \, \mathrm{d}k - \log \int_0^1 \varepsilon_{ik}^{\theta_i - 1} \, \mathrm{d}k.$$

As a result, $\log \varepsilon_i$ is equal to the cross-sectional average of $\log \varepsilon_{ik}$ to a first-order approximation.

Hence, market clearing for sectoral good i implies that $p_i y_i = p_i c_i + \sum_{j=1}^n a_{ji} p_j y_j \varepsilon_j$. Dividing both sides by nominal aggregate demand, PC, and using the fact that $p_i c_i = \beta_i PC$, we obtain

$$\lambda_i = \beta_i + \sum_{j=1}^n a_{ji}\varepsilon_j\lambda_j, \tag{B.25}$$

where $\lambda_i = p_i y_i / PC$ is the Domar weight of industry i.

Next, note that the representative household's budget constraint is given by

$$PC = wL + \sum_{i=1}^n \left(p_i y_i - \mathrm{mc}_i \int_0^1 y_{ik}\,\mathrm{d}k \right) = wL + \sum_{i=1}^n (1-\varepsilon_i) p_i y_i,$$

which implies that $PC = wL/(1 - \sum_{i=1}^n \lambda_i(1-\varepsilon_i))$. Furthermore, the household's optimal labor supply requires that $L^{1/\eta} = C^{-\gamma} w/P$. Therefore, solving for household's aggregate consumption and aggregate labor supply from the last two equations, we obtain

$$\begin{aligned} C &= (w/P)^{\frac{1+1/\eta}{\gamma+1/\eta}} \left(1 - \sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right)^{-\frac{1/\eta}{\gamma+1/\eta}}, \\ L &= (w/P)^{\frac{1-\gamma}{\gamma+1/\eta}} \left(1 - \sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right)^{\frac{\gamma}{\gamma+1/\eta}}. \end{aligned} \tag{B.26}$$

Plugging the above into (2), we can express the representative household's welfare as a function of nominal prices and the nominal wage as

$$\begin{aligned} W = {} & \frac{1}{1-\gamma} (w/P)^{\frac{(1-\gamma)(1+1/\eta)}{\gamma+1/\eta}} \left(1 - \sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right)^{-\frac{(1-\gamma)/\eta}{\gamma+1/\eta}} \\ & \times \left(1 - \frac{1-\gamma}{1+1/\eta}\left(1 - \sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right)\right), \end{aligned} \tag{B.27}$$

where ε_i is given by (B.24) and the Domar weights solve the system of equations in (B.25).

Welfare Loss. As our next step, we compare (B.27) to welfare under the first-best allocation to obtain an expression for the welfare loss as a function of nominal prices.

Recall that, in the flexible-price equilibrium, all firms in industry i set identical prices and charge no markups, that is, $\mathrm{mc}_i^* = p_{ik}^* = p_i^*$. Therefore, equation (B.24) implies that $\varepsilon_i^* = 1$ for all i. Plugging this into (B.27) implies that $W^* = \frac{\gamma+1/\eta}{(1-\gamma)(1+1/\eta)}(w/P^*)^{\frac{(1-\gamma)(1+1/\eta)}{\gamma+1/\eta}}$, where recall that, by assumption, $w = w^*$. Hence, we can rewrite (B.27) as

$$W = W^* (P/P^*)^{\frac{(\gamma-1)(1+1/\eta)}{\gamma+1/\eta}} \left(1 - \sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right)^{-\frac{(1-\gamma)/\eta}{\gamma+1/\eta}} \left(1 + \frac{1-\gamma}{\gamma+1/\eta}\sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right).$$

Similarly, we can use (B.26) to relate aggregate output in the sticky-price equilibrium to that in the flexible-price equilibrium:

$$C = C^*\big(P/P^*\big)^{-\frac{1+1/\eta}{\gamma+1/\eta}}\left(1-\sum_{i=1}^{n}\lambda_i(1-\varepsilon_i)\right)^{-\frac{1/\eta}{\gamma+1/\eta}}. \tag{B.28}$$

The juxtaposition of the last two equations then implies that welfare in the sticky- and flexible-price equilibria are related to one another as follows:

$$W = W^*\big(C/C^*\big)^{1-\gamma}\left(1+\frac{1-\gamma}{\gamma+1/\eta}\big(1-\big(C/C^*\big)^{-(1+\eta\gamma)}\big(P/P^*\big)^{-(1+\eta)}\big)\right). \tag{B.29}$$

Second-Order Approximations. We next derive log-quadratic approximations to equations (B.24), (B.28), and (B.29) around the economy's steady-state as $\delta \to 0$.

First, consider equation (B.24). Taking logarithms from both sides and using the fact that $\log \mathrm{mc}_i = \alpha \log w - \log z_i + \sum_{j=1}^{n} a_{ij}\log p_j$ implies that

$$\log\varepsilon_i = \sum_{j=1}^{n} a_{ij}\big(\log p_j - \log p_j^*\big) + (\theta_i - 1)\big(\log p_i - \log p_i^*\big) + \log\int_0^1 \big(p_{ik}/p_i^*\big)^{-\theta_i}\,\mathrm{d}k.$$

Consequently, to a second-order approximation,

$$\log\varepsilon_i = \sum_{j=1}^{n} a_{ij}\bar{e}_j - \bar{e}_i + \frac{1}{2}\sum_{j=1}^{n} a_{ij}(1-\theta_j)\vartheta_j + \frac{1}{2}(2\theta_i - 1)\vartheta_i + o\big(\delta^2\big), \tag{B.30}$$

where $\bar{e}_i$ and ϑ_i are the cross-sectional average and dispersion of pricing errors in industry i defined in (34) and (35), respectively, and we are using the fact that $\log p_j - \log p_j^* = \bar{e}_j + \frac{1}{2}(1-\theta_j)\vartheta_j + o(\delta^2)$.

We next derive a log-quadratic approximation to (B.28). Start with the observation that

$$\log\left(1-\sum_{i=1}^{n}\lambda_i(1-\varepsilon_i)\right)$$
$$= \sum_{i=1}^{n}\lambda_i\log\varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\lambda_i\log^2\varepsilon_i - \frac{1}{2}\left(\sum_{i=1}^{n}\lambda_i\log\varepsilon_i\right)^2 + o\big(\delta^2\big). \tag{B.31}$$

By (B.25), the vector of industry Domar weights is given by $\lambda = (\mathbf{I} - \mathbf{A}'\,\mathrm{diag}(\varepsilon))^{-1}\beta$. As a result,

$$\begin{aligned}
\lambda - \lambda^* &= \big(\mathbf{I} - \mathbf{A}'\,\mathrm{diag}(\varepsilon)\big)^{-1}\beta - \big(\mathbf{I} - \mathbf{A}'\big)^{-1}\beta \\
&= \big(\mathbf{I} - \mathbf{A}'\,\mathrm{diag}(\varepsilon)\big)^{-1}\mathbf{A}'\big(\mathrm{diag}(\varepsilon) - \mathbf{I}\big)\big(\mathbf{I} - \mathbf{A}'\big)^{-1}\beta \\
&= \mathbf{L}'\mathbf{A}'\,\mathrm{diag}(\log\varepsilon)\lambda^* + o(\delta) = \big(\mathbf{L}' - \mathbf{I}\big)\,\mathrm{diag}(\log\varepsilon)\lambda^* + o(\delta),
\end{aligned}$$

where $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ is the economy's Leontief inverse and we are using the fact that the vector of Domar weights under flexible prices is given by $\lambda^* = \mathbf{L}'\beta$. The above equation implies that, to a first-order approximation, the Domar weight of industry i in the sticky-price equilibrium is given by

$$\lambda_i = (1-\log\varepsilon_i)\lambda_i^* + \sum_{j=1}^n \ell_{ji}\lambda_j^* \log\varepsilon_j + o(\delta). \tag{B.32}$$

Putting the above equation together with (B.31) leads to

$$\log\left(1-\sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right) = \sum_{i=1}^n \lambda_i^*\log\varepsilon_i + \sum_{i=1}^n\sum_{j=1}^n \lambda_j^*\ell_{ji}\log\varepsilon_j\log\varepsilon_i$$
$$-\frac{1}{2}\sum_{i=1}^n \lambda_i^*\log^2\varepsilon_i - \frac{1}{2}\left(\sum_{i=1}^n \lambda_i^*\log\varepsilon_i\right)^2 + o(\delta^2).$$

Replacing $\log\varepsilon_i$ by its second-order approximation in (B.30), we obtain

$$\log\left(1-\sum_{i=1}^n \lambda_i(1-\varepsilon_i)\right) = -(\log P - \log P^*) + \frac{1}{2}\sum_{i=1}^n \lambda_i^*\theta_i\vartheta_i + \frac{1}{2}\sum_{i=1}^n \lambda_i^*\bar{e}_i^2$$
$$-\frac{1}{2}\sum_{i=1}^n \lambda_i^*\left(\sum_{j=1}^n a_{ij}\bar{e}_j\right)^2 - \frac{1}{2}\left(\sum_{i=1}^n \beta_i\bar{e}_i\right)^2 + o(\delta^2),$$

where we are using the fact that $\log P - \log P^* = \sum_{i=1}^n \beta_i\bar{e}_i + \frac{1}{2}\sum_{i=1}^n \beta_i(1-\theta_i)\vartheta_i + o(\delta^2)$. Consequently, we obtain the following second-order approximation to (B.28):

$$\log C - \log C^* = -\frac{1/\eta}{2(\gamma+1/\eta)}\left[\sum_{i=1}^n \lambda_i^*\theta_i\vartheta_i + \sum_{i=1}^n \lambda_i^*\left(\bar{e}_i^2 - \left(\sum_{j=1}^n a_{ij}\bar{e}_j\right)^2\right) - \left(\sum_{i=1}^n \beta_i\bar{e}_i\right)^2\right]$$
$$-\frac{1}{\gamma+1/\eta}(\log P - \log P^*) + o(\delta^2). \tag{B.33}$$

The above expression implies that, to a first-order approximation, the output gap is

$$\log C - \log C^* = -\frac{1}{\gamma+1/\eta}(\log P - \log P^*) + o(\delta). \tag{B.34}$$

Finally, we derive a log-quadratic approximation to equation (B.29) as $\delta\to 0$. We have

$$\begin{aligned}
&\log(W/W^*)\\
&\quad = (1-\gamma)(1+\eta)\\
&\quad\quad \times\Bigg[(\log C - \log C^*) + \frac{1}{\gamma+1/\eta}(\log P - \log P^*) - \frac{1}{2}\eta(\log C - \log C^*)^2\\
&\quad\quad -\frac{1}{2}\eta\frac{(1+\eta)^2}{(1+\eta\gamma)^2}(\log P - \log P^*)^2 - \frac{1+\eta}{\gamma+1/\eta}(\log C - \log C^*)(\log P - \log P^*)\Bigg]\\
&\quad\quad + o(\delta^2).
\end{aligned}$$

Replace for $\log P - \log P^*$ in the last two terms from its first-order approximation in (B.34) to obtain

$$\log(W/W^*) = (1-\gamma)(1+\eta)\left(\log C - \log C^* + \frac{1}{\gamma + 1/\eta}(\log P - \log P^*) - \frac{1}{2\eta}\Delta^2\right) + o(\delta^2),$$

where $\Delta^2 = (\log C - \log C^*)^2$ is the volatility of output gap. Using (B.33) to replace for the first term on the right-hand side above and using the fact that $\lambda_i^* = \beta_i + \sum_{j=1}^n \lambda_j^* a_{ji}$ implies that

$$\log(W/W^*) = -\frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{(\gamma+1/\eta)}\left[\sum_{i=1}^n \lambda_i^* \theta_i \vartheta_i + (\gamma + 1/\eta)\Delta^2 + \sum_{i=1}^n \lambda_i^* \operatorname{xvar}_i(\bar{e}_1, \dots, \bar{e}_n) + \operatorname{xvar}_0(\bar{e}_1, \dots, \bar{e}_n)\right] + o(\delta^2),$$

where $\operatorname{xvar}_i(\bar{e}_1, \dots, \bar{e}_n)$ is the cross-sectional dispersion of pricing errors of inputs from the point of view of industry i defined in (38). We make two final observations. First, the fact that $\lambda_i = \lambda_i^* + o(1)$ as $\delta \to 0$ implies that we can replace λ_i^* by λ_i in the above equation. Second, note that $W - W^* = W^* \log(W/W^*) + o(\delta^2)$ and $W^* = \frac{\gamma+1/\eta}{(1-\gamma)(1+1/\eta)} + O(\delta)$. The juxtaposition of these observations with the above equation then establishes (36).

Proof of Proposition 5

We start by stating and proving a lemma, which we will also use in the proof of Theorem 2. Statement (a) of the lemma establishes that even though in our model the monetary policy instrument is the nominal aggregate demand $m(z)$, as long as no industry is perfectly sticky, there is an isomorphism between setting the nominal aggregate demand and the nominal wage $w(z)$. Statement (b) of the lemma then provides conditions under which a policy can be implemented as a price-stabilization policy.

LEMMA B.1: *Suppose $\phi_i > 0$ for all i. Then, to a first-order approximation,*
(a) *an allocation is implementable by setting the nominal demand if and only if it is implementable by setting the nominal wage*;
(b) *if vector κ satisfies $\kappa'\alpha = 1$, then the nominal wage $\log w(z) = \sum_{i=1}^n \kappa_i \log z_i$ can be implemented by a price-stabilization policy of the form $\sum_{i=1}^n \psi_i \log z_i = 0$ for some vector $\psi = (\psi_1, \dots, \psi_n)'$.*

Proof of Part (a). It is sufficient to show that, as long as $\phi_i > 0$ for all industries i, there is a one-to-one correspondence between the nominal wage $w(z)$ and nominal aggregate demand $m(z)$ for all realizations of z. Let P^* and C^* denote, respectively, the consumption price index and aggregate output in the flexible-price equilibrium. Since $m = PC$, it is immediate that

$$\log m = \left(1 - \frac{1}{\gamma + 1/\eta}\right)(\log P - \log P^*) + \log P^* + \log C^* + o(\delta),$$

where we are using (B.34). Next, recall from Proposition 4 that industry-level nominal prices satisfy (30). Therefore, the vector of average pricing errors defined in (34) is given by

$$\bar{e} = \boldsymbol{\Phi}(\mathbf{I} - \mathbf{A}\boldsymbol{\Phi})^{-1}(\alpha \log w - \log z) - (\mathbf{I} - \mathbf{A})^{-1}(\alpha \log w - \log z) + o(\delta),$$

or equivalently,

$$\bar{e} = \mathbf{Q}(\mathbf{L} \log z - \mathbf{1} \log w) + o(\delta), \tag{B.35}$$

where $\mathbf{Q} = (\mathbf{I} - \boldsymbol{\Phi}\mathbf{A})^{-1}(\mathbf{I} - \boldsymbol{\Phi})$. Therefore,

$$\log m = \left(1 - \frac{1}{\gamma + 1/\eta}\right)\beta'\mathbf{Q}(\mathbf{L} \log z - \mathbf{1} \log w) + \log P^* + \log C^* + o(\delta),$$

where we are using the fact that $\log P - \log P^* = \beta'\bar{e} + o(\delta)$. It is also immediate to verify that, in the flexible-price equilibrium, the consumption price index and aggregate output are given by $\log P^* = \log w - \beta'\mathbf{L} \log z$ and $\log C^* = \frac{1+1/\eta}{\gamma+1/\eta}\beta'\mathbf{L} \log z$, respectively. As a result, to a first-order approximation, nominal wage and nominal aggregate demand are related to one another via the following relationship:

$$\begin{aligned} \log m = {} & \left[\left(1 - \frac{1}{\gamma + 1/\eta}\right)\beta'\mathbf{Q}\mathbf{L} - \frac{\gamma - 1}{\gamma + 1/\eta}\beta'\mathbf{L}\right] \log z \\ & + \left[1 - \left(1 - \frac{1}{\gamma + 1/\eta}\right)\beta'\mathbf{Q}\mathbf{1}\right] \log w + o(\delta). \end{aligned} \tag{B.36}$$

The above equation establishes a one-to-one correspondence between $w(z)$ and $m(z)$ as long as $(1 - \frac{1}{\gamma+1/\eta})\beta'\mathbf{Q}\mathbf{1} \neq 1$. The proof is therefore complete once we show that this condition is indeed satisfied. To this end, note that it is sufficient to show that $0 \leq \beta'\mathbf{Q}\mathbf{1} < 1$. The fact that $\beta'\mathbf{Q}\mathbf{1} \geq 0$ is a straightforward implication of the fact that $(\mathbf{I} - \boldsymbol{\Phi}\mathbf{A})^{-1}$ is an inverse M-matrix, and hence is elementwise nonnegative. To show that $\beta'\mathbf{Q}\mathbf{1} < 1$, note that

$$1 - \beta'\mathbf{Q}\mathbf{1} = \beta'\boldsymbol{\Phi}(\mathbf{I} - \mathbf{A}\boldsymbol{\Phi})^{-1}(\mathbf{I} - \mathbf{A})\mathbf{1} = \sum_{i=1}^{n}\sum_{j=1}^{n}\beta_i\phi_i h_{ij}\alpha_j,$$

where $\mathbf{H} = (\mathbf{I} - \mathbf{A}\boldsymbol{\Phi})^{-1}$. The fact that $\mathbf{H}$ is an inverse M-matrix guarantees that the right-hand side of the above equation is nonnegative. To show that it is in fact strictly positive, suppose to the contrary that it is equal to zero. This means that

$$\beta_i\phi_i h_{ij}\alpha_j = 0$$

for all pairs of industries i and j. But for any industry i, there exists at least one industry j (which may coincide with i) such that $\alpha_j > 0$ and $h_{ij} > 0$. This, coupled with the fact that $\sum_{i=1}^{n}\beta_i = 1$ and the assumption that $\phi_i > 0$ for all i, leads to a contradiction. Therefore, it must be the case that $\beta'\mathbf{Q}\mathbf{1} < 1$, which completes the proof.

Proof of Part (b). Let vector $\kappa = (\kappa_1, \dots, \kappa_n)'$ satisfy $\kappa'\alpha = 1$. We show that stabilizing the price index $\sum_{i=1}^n \psi_i \log p_i$ with weights given by

$$\psi' = \kappa'\mathbf{\Phi}^{-1}(\mathbf{I} - \mathbf{\Phi}\mathbf{A}) \tag{B.37}$$

induces a nominal wage given by $\log w = \sum_{i=1}^n \kappa_i \log z_i + o(\delta)$. Start by noting that the juxtaposition of $\psi' \log p = 0$ and equation (30) implies that $\psi'(\mathbf{I} - \mathbf{\Phi}\mathbf{A})^{-1}\mathbf{\Phi}(\mathbf{I} - \mathbf{A})(\mathbf{1}\log w - \mathbf{L}\log z) = o(\delta)$. Consequently,

$$\log w = \frac{1}{\psi'(\mathbf{I} - \mathbf{\Phi}\mathbf{A})^{-1}\mathbf{\Phi}(\mathbf{I} - \mathbf{A})\mathbf{1}} \psi'(\mathbf{I} - \mathbf{\Phi}\mathbf{A})^{-1}\mathbf{\Phi}\log z + o(\delta).$$

Replacing for ψ from (B.37) into the above implies that $\log w = \frac{1}{\kappa'\alpha}\kappa' \log z + o(\delta)$. The assumption that $\kappa'\alpha = 1$ then implies that $\log w = \sum_{i=1}^n \kappa_i \log z_i + o(\delta)$.

Proof of Proposition 5. With Lemma B.1 in hand, we are now ready to prove the proposition. First, note that the identity $PC = m$ implies that degree of monetary nonneutrality satisfies

$$\Xi = 1 - \frac{\mathrm{d}\log P}{\mathrm{d}\log m} = 1 - \frac{\mathrm{d}\log P}{\mathrm{d}\log w}\frac{\mathrm{d}\log w}{\mathrm{d}\log m}.$$

Since $\log P = \sum_{i=1}^n \beta_i \log p_i$, equation (30) implies that $\mathrm{d}\log P/\mathrm{d}\log w = \sum_{i=1}^n \beta_i\phi_i\rho_i = \rho_0$, where the second equality follows from the definition of ρ_0 in (32). Thus, using (B.36), we obtain

$$\Xi = 1 - \rho_0\left[1 - \left(1 - \frac{1}{\gamma + 1/\eta}\right)\beta'\mathbf{Q}\mathbf{1}\right]^{-1},$$

where $\mathbf{Q} = (\mathbf{I} - \mathbf{\Phi}\mathbf{A})^{-1}(\mathbf{I} - \mathbf{\Phi})$. Finally, noting that $\beta'\mathbf{Q}\mathbf{1} = \beta'(\mathbf{1} - \mathbf{\Phi}\rho) = 1 - \rho_0$ establishes (33).

Proof of Theorem 2

As a first observation, we note that the independence assumption imposed on the noise shocks $\epsilon_{ij,k}$ implies that aggregate uncertainty in this economy is solely driven by the productivity shocks $z = (z_1, \dots, z_n)$. As a result, without loss of generality, we can restrict our attention to monetary policies of the form $m(z)$ that only depend on the productivity shocks, as opposed to the entire state of the economy, $s = (z, \omega)$.

By Lemma B.1, as long as $\phi_i > 0$ for all i, any allocation that is implementable by setting nominal aggregate demand, $m(z)$, is also implementable by setting the nominal wage, $w(z)$. Therefore, to determine the optimal policy, we first characterize how the wage should optimally respond to productivity shocks. Specifically, we characterize the vector of optimal weights $\kappa = (\kappa_1, \dots, \kappa_n)'$ in $\log w = \sum_{j=1}^n \kappa_j \log z_j$ that minimizes the expected welfare loss in (36). We then show that the price-stabilization policy with weights given by (39) implements such a nominal wage.

To calculate the expected welfare loss in (36), we first determine the cross-sectional average and dispersion of pricing errors in each industry, defined in equations (34) and (35), respectively. In the proof of Lemma B.1, we already established that the vector of average pricing errors is given by (B.35). To determine the within-industry dispersion of pricing errors, note that all firm-level nominal prices within the same industry coincide with one another in the flexible-price equilibrium. Therefore, the dispersion of pricing errors in industry i satisfies

$$\vartheta_i = \int_0^1 e_{ik}^2 \, \mathrm{d}k - \left(\int_0^1 e_{ik} \, \mathrm{d}k\right)^2 = \int_0^1 (\log p_{ik} - \log p_i)^2 \, \mathrm{d}k + o(\delta^2).$$

Furthermore, recall from the proof of Proposition 4 that the nominal price of firm k in industry i in the sticky-price equilibrium is given by (B.22), which implies that

$$\log p_{ik} = \frac{\phi_{ik}}{\phi_i} \sum_{j=1}^n b_{ij} \omega_{ij,k} + o(\delta),$$

where matrix $\mathbf{B}$ is given by (B.23). Consequently,

$$\log p_{ik} - \log p_i = \left(\frac{\phi_{ik}}{\phi_i} - 1\right) \sum_{j=1}^n b_{ij} \log z_i + \frac{\phi_{ik}}{\phi_i} \sum_{j=1}^n b_{ij} \epsilon_{ij,k} + o(\delta).$$

Therefore, the expected cross-sectional dispersion of pricing errors in industry i is given by

$$\begin{aligned}\mathbb{E}[\vartheta_i] &= \frac{\delta^2}{\phi_i^2} \sum_{j=1}^n b_{ij}^2 \left(\sigma_z^2 \int_0^1 (\phi_{ik} - \phi_i)^2 \, \mathrm{d}k + \int_0^1 \phi_{ik}^2 \sigma_{ik}^2 \, \mathrm{d}k\right) + o(\delta^2) \\ &= \sigma_z^2 \frac{\delta^2}{\phi_i^2} \sum_{j=1}^n b_{ij}^2 \left(\int_0^1 (\phi_{ik} - \phi_i)^2 \, \mathrm{d}k + \int_0^1 \phi_{ik}(1 - \phi_{ik}) \, \mathrm{d}k\right) + o(\delta^2),\end{aligned}$$

where the second equality is a simple consequence of the definition of ϕ_{ik} in (25). Hence,

$$\mathbb{E}[\vartheta_i] = \sigma_z^2 \delta^2 \left(\frac{1 - \phi_i}{\phi_i}\right) \sum_{j=1}^n b_{ij}^2 + o(\delta^2).$$

With the expected cross-sectional average and dispersion of pricing errors in hand, we now minimize the expected welfare loss by optimizing over the vector $\kappa = (\kappa_1, \ldots, \kappa_n)$, where recall that $\log w = \sum_{j=1}^n \kappa_j \log z_j$. Taking expectations from both sides of (36), differentiating it with respect to κ_s, and setting it equal to zero, we obtain

$$\begin{aligned}&\sigma_z^2 \delta^2 \sum_{i=1}^n \lambda_i \theta_i \left(\frac{1 - \phi_i}{\phi_i}\right) \sum_{j=1}^n b_{ij} \frac{\mathrm{d}b_{ij}}{\mathrm{d}\kappa_s} + \frac{1}{\gamma + 1/\eta} \sum_{i=1}^n \sum_{j=1}^n \beta_i \beta_j \mathbb{E}\left[\bar{e}_i \frac{\mathrm{d}\bar{e}_j}{\mathrm{d}\kappa_s}\right] \\ &\quad + \sum_{i=0}^n \lambda_i \left(\sum_{j=1}^n a_{ij} \mathbb{E}\left[\bar{e}_j \frac{\mathrm{d}\bar{e}_j}{\mathrm{d}\kappa_s}\right] - \sum_{j=1}^n \sum_{r=1}^n a_{ij} a_{ir} \mathbb{E}\left[\bar{e}_r \frac{\mathrm{d}\bar{e}_j}{\mathrm{d}\kappa_s}\right]\right) = 0,\end{aligned}$$

with the convention that $\lambda_0 = 1$ and $a_{0j} = \beta_j$ for all j. To simplify the above, note that (B.35) implies that $\mathrm{d}\bar{e}_j / \mathrm{d}\kappa_s = -\log z_s \sum_{r=1}^n q_{jr}$, while equation (B.23) implies that

$\mathrm{d}b_{ij}/\mathrm{d}\kappa_s = 0$ if $j \neq s$. As a result,

$$\sum_{i=1}^n \lambda_i \theta_i \left(\frac{1-\phi_i}{\phi_i}\right) \frac{\mathrm{d}b_{is}}{\mathrm{d}\kappa_s} b_{is} - \frac{1}{\gamma + 1/\eta}\left(\sum_{i=1}^n \sum_{j=1}^n \beta_i q_{ij}(\ell_{js} - \kappa_s)\right)\left(\sum_{i=1}^n \sum_{j=1}^n \beta_i q_{ij}\right)$$
$$- \sum_{i=0}^n \sum_{j=1}^n \lambda_i a_{ij}\left(\sum_{r=1}^n q_{jr}(\ell_{rs} - \kappa_s)\right)\left(\sum_{m=1}^n q_{jm}\right)$$
$$+ \sum_{i=0}^n \lambda_i \left(\sum_{j=1}^n \sum_{r=1}^n a_{ij} q_{jr}\right)\left(\sum_{j=1}^n \sum_{r=1}^n a_{ij} q_{jr}(\ell_{rs} - \kappa_s)\right) = 0.$$

Since the above first-order condition has to hold for all s, it can be rewritten in matrix form as

$$\Bigg[\lambda' \operatorname{diag}(\theta)(\mathbf{I} - \mathbf{\Phi})\mathbf{\Phi}^{-1}\big(\mathbf{I} - \operatorname{diag}(\mathbf{Q1})\big)(\mathbf{I} - \mathbf{Q}) + \frac{1}{\gamma + 1/\eta}\big(\beta'\mathbf{Q1}\big)\beta'\mathbf{Q}$$
$$+ \big(\lambda'\mathbf{A} + \beta'\big) \operatorname{diag}(\mathbf{Q1})\mathbf{Q} - \lambda' \operatorname{diag}(\mathbf{AQ1})\mathbf{AQ} - \big(\beta'\mathbf{Q1}\big)\beta'\mathbf{Q}\Bigg]\big(\mathbf{L} - \mathbf{1}\kappa'\big) = 0,$$

where we are using the fact that matrix $\mathbf{B}$ defined in (B.23) satisfies $\mathbf{B} = (\mathbf{I} - \mathbf{Q})(\mathbf{1}\kappa' - \mathbf{L})$. Solving for κ then implies that the vector of weights κ in $\log w = \sum_{i=1}^n \kappa_i \log z_i$ that minimizes the expected welfare loss is given by $\kappa' = \iota'/(\iota'\alpha)$, where

$$\iota' = \lambda' \operatorname{diag}(\theta)(\mathbf{I} - \mathbf{\Phi})\mathbf{\Phi}^{-1}\big(\mathbf{I} - \operatorname{diag}(\mathbf{Q1})\big)(\mathbf{I} - \mathbf{Q})\mathbf{L} + \frac{1}{\gamma + 1/\eta}\big(\beta'\mathbf{Q1}\big)\beta'\mathbf{QL}$$
$$+ \lambda'\mathbf{A} \operatorname{diag}(\mathbf{Q1})\mathbf{QL} - \lambda' \operatorname{diag}(\mathbf{AQ1})\mathbf{AQL}$$
$$+ \beta' \operatorname{diag}(\mathbf{Q1})\mathbf{QL} - \big(\beta'\mathbf{Q1}\big)\beta'\mathbf{QL}. \tag{B.38}$$

Having determined the nominal wage that minimizes expected welfare loss, we next determine the price-stabilization policy that implements such a nominal wage. First, note that since $\kappa' = \iota'/(\iota'\alpha)$, we have $\kappa'\alpha = 1$. As a result, we can apply statement (b) of Lemma B.1, which guarantees that the optimal nominal wage $\log w = \sum_{i=1}^n \kappa_i \log z_i$ can be implemented by stabilizing the price index $\sum_{i=1}^n \psi_i^* \log p_i$, where the industry weights are given by (B.37), that is, $\psi^{*\prime} = \kappa'\mathbf{\Phi}^{-1}(\mathbf{I} - \mathbf{\Phi A})$. Hence,

$$\psi^{*\prime} = \lambda' \operatorname{diag}(\theta)\big(\mathbf{I} - \operatorname{diag}(\mathbf{Q1})\big)\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big) + \frac{1}{\gamma + 1/\eta}\big(\beta'\mathbf{Q1}\big)\beta'\mathbf{L}\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big)$$
$$+ \big(\lambda'\mathbf{A} \operatorname{diag}(\mathbf{Q1}) - \lambda' \operatorname{diag}(\mathbf{AQ1})\mathbf{A}\big)\mathbf{L}\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big)$$
$$+ \big(\beta' \operatorname{diag}(\mathbf{Q1}) - \big(\beta'\mathbf{Q1}\big)\beta'\big)\mathbf{L}\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big), \tag{B.39}$$

where we are using matrix identities $\mathbf{QL\Phi}^{-1}(\mathbf{I} - \mathbf{\Phi A}) = \mathbf{L}(\mathbf{\Phi}^{-1} - \mathbf{I})$ and $(\mathbf{I} - \mathbf{Q})\mathbf{L\Phi}^{-1}(\mathbf{I} - \mathbf{\Phi A}) = \mathbf{I}$. Next, note that the definition of ρ in (28) implies that $\mathbf{Q1} = \mathbf{1} - \mathbf{\Phi}\rho$. Therefore,

$$\psi^{*\prime} = \lambda'(\mathbf{I} - \mathbf{\Phi}) \operatorname{diag}(\theta) \operatorname{diag}(\rho) + \left(\frac{1 - \beta'\mathbf{\Phi}\rho}{\gamma + 1/\eta}\right)\lambda'\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big)$$

$$+ \lambda'\big(\mathbf{A} - \mathbf{A\Phi}\,\mathrm{diag}(\rho) - \mathrm{diag}(\mathbf{A1} - \mathbf{A\Phi}\rho)\mathbf{A}\big)\mathbf{L}\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big)$$
$$+ \big((\beta'\mathbf{\Phi}\rho)\beta' - \beta'\mathbf{\Phi}\,\mathrm{diag}(\rho)\big)\mathbf{L}\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big).$$

Since $\mathbf{A\Phi}\rho = \rho - (\mathbf{I} - \mathbf{A})\mathbf{1}$, we have $\mathbf{A} - \mathbf{A\Phi}\,\mathrm{diag}(\rho) - \mathrm{diag}(\mathbf{A1})\mathbf{A} + \mathrm{diag}(\mathbf{A\Phi}\rho)\mathbf{A} = \mathrm{diag}(\rho)\mathbf{A} - \mathbf{A\Phi}\,\mathrm{diag}(\rho)$. As a result,

$$\psi^{*\prime} = \lambda'(\mathbf{I} - \mathbf{\Phi})\,\mathrm{diag}(\theta)\,\mathrm{diag}(\rho) + \left(\frac{1 - \beta'\mathbf{\Phi}\rho}{\gamma + 1/\eta}\right)\lambda'\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big)$$
$$+ \lambda'\big(\mathrm{diag}(\rho)\mathbf{A} - \mathbf{A\Phi}\,\mathrm{diag}(\rho)\big)\mathbf{L}\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big) + \big((\beta'\mathbf{\Phi}\rho)\beta' - \beta'\mathbf{\Phi}\,\mathrm{diag}(\rho)\big)\mathbf{L}\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big).$$

Noting that $\lambda'\mathbf{A} = \lambda' - \beta'$ and $\rho_0 = \beta'\mathbf{\Phi}\rho$, we can simplify the above equation as

$$\psi^{*\prime} = \lambda'(\mathbf{I} - \mathbf{\Phi})\,\mathrm{diag}(\theta)\,\mathrm{diag}(\rho) + \left(\frac{1 - \rho_0}{\gamma + 1/\eta}\right)\lambda'\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big)$$
$$+ \lambda'\big((\mathbf{I} - \mathbf{\Phi})\,\mathrm{diag}(\rho)\mathbf{L} + \rho_0\mathbf{I} - \mathrm{diag}(\rho)\big)\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big).$$

Consequently, the weight on the price of industry s in the optimal price-stabilization policy satisfies

$$\psi_s^* = (1/\phi_s - 1)\left[\lambda_s\phi_s\theta_s\rho_s + \lambda_s\left(\frac{1 - \rho_0}{\gamma + 1/\eta}\right) + \sum_{i=1}^n (1 - \phi_i)\lambda_i\rho_i\ell_{is} + (\rho_0 - \rho_s)\lambda_s\right],$$

an expression that coincides with (39).

Proof of Proposition 7

Since i and j are upstream symmetric, it is immediate that $\rho_i = \rho_j$. Furthermore, the fact that they are downstream symmetric implies that they have identical Domar weights, that is, $\lambda_i = \lambda_j$. As a result, equations (40)–(42) in Theorem 2 imply that if $\phi_i < \phi_j$, then $\psi_i^{\text{o.g.}} > \psi_j^{\text{o.g.}}$, $\psi_i^{\text{across}} > \psi_j^{\text{across}}$, and $\psi_i^{\text{within}} > \psi_j^{\text{within}}$. Putting the three inequalities together then guarantees that $\psi_i^* > \psi_j^*$.

Proof of Proposition 8

Since i and j are downstream symmetric, they have identical Domar weights, that is, $\lambda_i = \lambda_j = \lambda$. In addition, recall that, by assumption, $\phi_i = \phi_j = \phi$. Therefore, equation (40) implies that $\psi_i^{\text{o.g.}} = \psi_j^{\text{o.g.}}$. Furthermore, equation (42) implies

$$\psi_i^{\text{across}} - \psi_j^{\text{across}} = (1/\phi - 1)\sum_{s=1}^n (1 - \phi_s)\lambda_s\rho_s(\ell_{si} - \ell_{sj}) = 0,$$

where once again we are using the assumptions that i and j are downstream symmetric and that $\phi_i = \phi_j = \phi$. Finally, equation (41) and the assumption that $\theta_i = \theta_j = \theta$ implies

$$\psi_i^{\text{within}} - \psi_j^{\text{within}} = (1 - \phi)\lambda\theta(\rho_i - \rho_j).$$

Thus, if $\rho_i > \rho_j$, then $\psi_i^{\text{within}} > \psi_j^{\text{within}}$, and hence, $\psi_i^* > \psi_j^*$.

Proof of Proposition 9

By assumption, $\theta_i = \theta_j$, $\phi_i = \phi_j$, and $\lambda_i = \lambda_j$. Furthermore, the assumption that i and j are upstream symmetric implies that $\rho_i = \rho_j$. Therefore, equations (40) and (41) in Theorem 2 imply that $\psi_i^{\text{o.g.}} = \psi_j^{\text{o.g.}}$ and $\psi_i^{\text{within}} = \psi_j^{\text{within}}$. Turning to the dimension of policy targeting across-industry misallocation, equation (42) implies

$$\psi_i^{\text{across}} - \psi_j^{\text{acorss}} = (1/\phi - 1)\sum_{s=1}^{n}(1-\phi_s)\lambda_s\rho_s(\ell_{si} - \ell_{sj}),$$

where once again we are using the fact that $\phi_i = \phi_j = \phi$. It is now immediate that $\psi_i^* > \psi_j^*$ if and only if inequality (43) is satisfied.

Proof of Proposition 10

Suppose $\theta_i = \theta_j = \theta$ and $\phi_i = \phi_j = \phi < 1$. Also suppose industry j is the sole supplier of industry i and i is the sole customer of j. This implies that both industries have identical steady-state Domar weights. Therefore, by (40), $\psi_i^{\text{o.g.}} = \psi_j^{\text{o.g.}}$.

The fact that j is the sole supplier of i also implies that $\rho_i = \phi_j\rho_j < \rho_j$, where we are using the definition of upstream flexbility in (28). As a result, equation (41) implies that, in the optimal policy, $\psi_i^{\text{within}} < \psi_j^{\text{within}}$.

Finally, consider the component of optimal policy corresponding to interindustry misallocation. Since i and j have identical Domar weights, equation (42) implies that

$$\psi_i^{\text{across}} - \psi_j^{\text{across}} = (1/\phi - 1)\left((\rho_j - \rho_i)\lambda_j + \sum_{s=1}^{n}(1-\phi_s)\lambda_s\rho_s(\ell_{si} - \ell_{sj})\right).$$

Furthermore, the assumption that j is the sole supplier of i and i is the sole customer of j implies that $\ell_{sj} = \ell_{si} + \mathbb{I}_{\{s=j\}}$ for all s. As a result,

$$\psi_i^{\text{across}} - \psi_j^{\text{across}} = (1/\phi - 1)\lambda_j(\phi_j\rho_j - \rho_i).$$

Now the fact that $\rho_i = \phi_j\rho_j$ guarantees that $\psi_i^{\text{across}} = \psi_j^{\text{across}}$.

Proof of Proposition A.1

Recall from the proof of Lemma 2 that the nominal price set by firm k in industry i is given by $\log p_{ik} = \mathbb{E}_{ik}[\log \mathrm{mc}_i] + o(\delta)$ as $\delta \to 0$. Therefore, when the production function of firms in industry i is given by (19),

$$\log p_{ik} = \alpha_i\mathbb{E}_{ik}[\log w] - \mathbb{E}_{ik}[\log z_i] + \sum_{j=1}^{n} a_{ij}\mathbb{E}_{ik}[\log p_j] + o(\delta),$$

where p_j is the nominal price of the sectoral good produced by industry j. Integrating both sides of this equation over the set of all firms in industry i implies that

$$\log p_i = \alpha_i\int_0^1 \mathbb{E}_{ik}[\log w]\,\mathrm{d}k - \int_0^1 \mathbb{E}_{ik}[\log z_i]\,\mathrm{d}k + \sum_{j=1}^{n} a_{ij}\int_0^1 \mathbb{E}_{ik}[\log p_j]\,\mathrm{d}k + o(\delta).$$

Since a similar expression has to hold for the nominal price of all industries j, iterating on the above equation implies that

$$\log p_i = \sum_{r=1}^{\infty} \bar{\mathbb{E}}_i^{(r)}[\alpha \log w] - \sum_{r=1}^{\infty} \bar{\mathbb{E}}_i^{(r)}[\log z] + o(\delta), \tag{B.40}$$

where $\bar{\mathbb{E}}_i^{(r)}[\cdot]$ is defined recursively in (A.1). This, coupled with the fact that $PC = m$ and $\log P = \sum_{i=1}^{n} \beta_i \log p_i$, implies that log aggregate output is given by

$$\log C = \sum_{r=1}^{\infty}\sum_{i=1}^{n} \beta_i \bar{\mathbb{E}}_i^{(r)}[\log z] + \left(\log m - \sum_{r=1}^{\infty}\sum_{i=1}^{n} \beta_i \bar{\mathbb{E}}_i^{(r)}[\alpha \log w]\right) + o(\delta).$$

Noting that $m = w$ when $\gamma = 1$ and $\eta \to \infty$ then establishes (A.2).

Proof of Proposition A.2

As a first observation, note that the log output of industry i is equal to $\log y_i = \log m - \log p_i + \log \lambda_i$, where λ_i is i's Domar weight. Using (B.32) to obtain a first-order approximation for $\log \lambda_i$ leads to

$$\log y_i = \log m - \log p_i + \log \lambda_i^* - \log \varepsilon_i + \frac{1}{\lambda_i^*}\sum_{j=1}^{n} \ell_{ji}\lambda_j^* \log \varepsilon_j + o(\delta).$$

Rewriting the above in vector form, we get

$$\log y = \mathbf{1}\log w - \log p + \log \lambda^* - \left(\mathbf{I} - \mathbf{\Lambda}^{*-1}\mathbf{L}'\mathbf{\Lambda}^*\right)\log \varepsilon + o(\delta),$$

where $\mathbf{\Lambda}^* = \mathrm{diag}(\lambda_1^*, \dots, \lambda_n^*)$ is a diagonal matrix with diagonal elements given by industries' flexible-price Domar weights and we are using the fact that $m = w$ when $\gamma = 1$ and $\eta \to \infty$. From the above equation, it is immediate that the vector of industry-level log output under flexible-prices is given by $\log y^* = \log w\mathbf{1} - \log p^* + \log \lambda^* + o(\delta)$. Consequently,

$$\log y = \log y^* - \left(\log p - \log p^*\right) - \left(\mathbf{I} - \mathbf{\Lambda}^{-1}\mathbf{L}'\mathbf{\Lambda}\right)\log \varepsilon + o(\delta),$$

where we are using the fact that $\lambda_i = \lambda_i^* + o(1)$. Equation (B.30) implies that $\log \varepsilon = -(\mathbf{I} - \mathbf{A})(\log p - \log p^*) + o(\delta)$, or equivalently, $\log p - \log p^* = -\mathbf{L}\log \varepsilon + o(\delta)$. Therefore,

$$\log y = \log y^* + \left(\mathbf{A} + \mathbf{\Lambda}^{-1}\mathbf{L}'\mathbf{\Lambda}(\mathbf{I} - \mathbf{A})\right)\mathbf{L}\log \varepsilon + o(\delta). \tag{B.41}$$

It is therefore sufficient to characterize $\log \varepsilon$ in terms of model primitives. To this end, first recall that $\log \varepsilon = -(\mathbf{I} - \mathbf{A})(\log p - \log p^*) + o(\delta)$. Furthermore, note that log nominal prices in the sticky and flexible-price equilibria are given by (B.40) and $\log p_i^* = \log m - \sum_{j=1}^{n} \ell_{ij} \log z_j$, respectively. Consequently,

$$\log \varepsilon = (\mathbf{I} - \mathbf{A})\left(\sum_{r=1}^{\infty} \bar{\mathbb{E}}^{(r)}[\log z] - \mathbf{L}\log z\right) + (\mathbf{I} - \mathbf{A})\left(\mathbf{1}\log m - \sum_{r=1}^{\infty} \bar{\mathbb{E}}^{(r)}[\alpha \log m]\right) + o(\delta),$$

where $\bar{\mathbb{E}}^{(r)}[t]$ is a vector whose i-th element is given by $\bar{\mathbb{E}}_i^{(r)}[t]$. Plugging the above into (B.41) completes the proof.

Proof of Proposition A.3

As in the proof of Theorem 2, we first determine the optimal policy by characterizing how nominal wage should optimally respond to productivity shocks. We then determine the price-stabilization policy that implements such a nominal wage.

Recall from Proposition 6 that nominal rigidities result in a welfare loss that can be approximated by (36) to a second-order approximation as $\delta \to 0$, where $\bar{e}_i$ and ϑ_i are defined in (34) and (35) and denote the cross-sectional average and cross-sectional dispersion of pricing errors in industry i, respectively. Therefore, as a first step, we determine $\bar{e}_i$ and ϑ_i in terms of the realized productivity shocks and the nominal wage. To this end, as in (B.20) and (B.21), let $\log w = \kappa' \log z + o(\delta)$ and $\log p = \mathbf{B} \log z + o(\delta)$ denote the log-linearization of, respectively, the nominal wage and the vector of nominal prices of sectoral goods as $\delta \to 0$. Furthermore, recall from Lemma 2 that, to a first-order approximation, the nominal price set by firm k in industry i is given by equation (27). Integrating both sides of (27) over all firms k in industry i implies that

$$\log p_i = \phi_i \alpha_i \log w - \phi_i \log z_i + \phi_i \sum_{j=1}^{n} a_{ij} \log p_j + o(\delta),$$

where we are using the assumption that a fraction ϕ_i of firms industry i receive perfectly informative signals about the realization of the shocks, while the remainder $1 - \phi_i$ fraction receive no information at all. Writing the above equation in matrix form and using (B.20) and (B.21) leads to $\mathbf{B} = \mathbf{\Phi}(\alpha\kappa' - \mathbf{I} + \mathbf{AB})$. Solving for matrix $\mathbf{B}$ therefore implies that the vector of log nominal prices is given by

$$\log p = \mathbf{B} \log z + o(\delta), \tag{B.42}$$

where

$$\mathbf{B} = (\mathbf{I} - \mathbf{\Phi A})^{-1} \mathbf{\Phi}(\mathbf{I} - \mathbf{A})\big(\mathbf{1}\kappa' - \mathbf{L}\big). \tag{B.43}$$

Equations (B.42) and (B.43) additionally imply that, in the absence of nominal rigidities, the vector of log nominal prices is $\log p^* = (\mathbf{1}\kappa' - \mathbf{L}) \log z + o(\delta)$. Therefore, the vector of cross-sectional average of pricing errors, defined in (34), is given by

$$\bar{e} = \log p - \log p^* = \mathbf{Q}\big(\mathbf{L} - \mathbf{1}\kappa'\big) \log z + o(\delta), \tag{B.44}$$

where $\mathbf{Q} = (\mathbf{I} - \mathbf{\Phi A})^{-1}(\mathbf{I} - \mathbf{\Phi})$. Next, we obtain the expression for cross-sectional dispersion of pricing errors within each industry. Since the marginal cost of firms in industry i is given by $\log \mathrm{mc}_i = \alpha_i \log w - \log z_i + \sum_{i=1}^{n} a_{ij} \log p_j$, equation (B.42) implies that

$$\log \mathrm{mc} = \big(\alpha\kappa' - \mathbf{I} + \mathbf{AB}\big) \log z + o(\delta) = \mathbf{\Phi}^{-1} \log p + o(\delta). \tag{B.45}$$

Furthermore, the assumption that a fraction ϕ_i of firms in industry i can set their prices flexibly implies that the cross-sectional dispersion of pricing errors in industry i is equal to $\vartheta_i = \phi_i(\log \mathrm{mc}_i - \log p_i)^2 + (1 - \phi_i)(\log p_i)^2 = (1/\phi_i - 1)(\log p_i)^2 + o(\delta^2)$, where the second equality follows from (B.45). Consequently, using (B.42), we obtain

$$\mathbb{E}[\vartheta_i] = \delta^2 (1/\phi_i - 1) b_i' \mathbf{\Sigma} b_i + o\big(\delta^2\big), \tag{B.46}$$

where b_i' is the ith row of matrix $\mathbf{B}$ in (B.42) and $\mathbf{\Sigma}$ denotes the variance-covariance matrix of log productivity shocks.

With the expressions for the cross-sectional average and dispersion of pricing errors in hand, we next turn to determining the expected welfare loss under an arbitrary policy. Recall from Proposition 6 that, to a second-order approximation as $\delta \to 0$, welfare loss is given by (36). Therefore, using (B.46), we can write the expected welfare loss as

$$\begin{aligned}\mathbb{E}\big[W - W^*\big] = -\frac{1}{2}\bigg[&\delta^2 \lambda' \operatorname{diag}(\theta)\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big) \operatorname{diag}\big(\mathbf{B\Sigma B}'\big) \\ &+ \frac{1}{\gamma + 1/\eta}\beta' \mathbb{E}\big[\bar{e}\bar{e}'\big]\beta + \big(\lambda'\mathbf{A} + \beta'\big) \operatorname{diag}\big(\mathbb{E}\big[\bar{e}\bar{e}'\big]\big) \\ &- \lambda' \operatorname{diag}\big(\mathbf{A}\mathbb{E}\big[\bar{e}\bar{e}'\big]\mathbf{A}'\big) - \beta'\mathbb{E}\big[\bar{e}\bar{e}'\big]\beta\bigg].\end{aligned}$$

Note that equation (B.44) implies that $\mathrm{d}\bar{e}/\mathrm{d}\kappa_s = -\mathbf{Q1} \log z_s$. Therefore, differentiating the expected welfare loss with respect to vector κ and setting it equal to zero implies that

$$\begin{aligned}&\mathbf{\Sigma}\boldsymbol{B}'\big(\mathbf{I} - \operatorname{diag}(\mathbf{Q1})\big)\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big) \operatorname{diag}(\theta)\lambda - \frac{1}{\gamma + 1/\eta}\mathbf{\Sigma}\big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}'\beta\beta'\mathbf{Q1} \\ &\quad - \mathbf{\Sigma}\big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}' \operatorname{diag}(\mathbf{Q1})\big(\mathbf{A}'\lambda + \beta\big) + \mathbf{\Sigma}\big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}'\mathbf{A}' \operatorname{diag}(\mathbf{AQ1})\lambda \\ &\quad + \mathbf{\Sigma}\big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}'\beta\beta'\mathbf{Q1} = 0.\end{aligned}$$

Multiplying both sides of the above equation by $\mathbf{\Sigma}^{-1}$ from the left and replacing for $\mathbf{B}'$ in the first term from (B.43) implies that

$$\begin{aligned}&-\big(\mathbf{L}' - \kappa\mathbf{1}'\big)\big(\mathbf{I} - \mathbf{Q}'\big)\big(\mathbf{I} - \operatorname{diag}(\mathbf{Q1})\big)\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big) \operatorname{diag}(\theta)\lambda - \frac{1}{\gamma + 1/\eta}\big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}'\beta\beta'\mathbf{Q1} \\ &\quad - \big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}' \operatorname{diag}(\mathbf{Q1})\big(\mathbf{A}'\lambda + \beta\big) + \big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}'\mathbf{A}' \operatorname{diag}(\mathbf{AQ1})\lambda \\ &\quad + \big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}'\beta\beta'\mathbf{Q1} = 0.\end{aligned}$$

Solving for κ then implies that the vector of weights in the optimal wage-setting policy is given by $\kappa' = \iota'/(\iota'\alpha)$, where

$$\begin{aligned}\iota' = {}&\lambda' \operatorname{diag}(\theta)(\mathbf{I} - \mathbf{\Phi})\mathbf{\Phi}^{-1}\big(\mathbf{I} - \operatorname{diag}(\mathbf{Q1})\big)(\mathbf{I} - \mathbf{Q})\mathbf{L} + \frac{1}{\gamma + 1/\eta}\big(\beta'\mathbf{Q1}\big)\beta'\mathbf{QL} \\ &+ \lambda'\mathbf{A} \operatorname{diag}(\mathbf{Q1})\mathbf{QL} - \lambda' \operatorname{diag}(\mathbf{AQ1})\mathbf{AQL} + \beta' \operatorname{diag}(\mathbf{Q1})\mathbf{QL} - \big(\beta'\mathbf{Q1}\big)\beta'\mathbf{QL}.\end{aligned}$$

The above expression is identical to equation (B.38). Therefore, the optimal price-stabilization policy coincides with the policy in (39).

Proof of Proposition A.4

We prove this result by establishing that the policy that minimizes the welfare loss (36) (i.e., the optimal policy) coincides with the policy that minimizes the volatility of the target price index $\sum_{i=1}^n \psi_i^* \log p_i$, with weights given by (39). Note that since $\gamma = 1$ and $\eta \to \infty$, the nominal aggregate demand, m, is equal to the nominal wage, w. As a result, without loss of generality, we can parameterize the set of policies by vector $\kappa = (\kappa_1, \dots, \kappa_n)'$, where $\log w = \sum_{i=1}^n \kappa_i \hat{\omega}_i$ and $\hat{\omega}_i$ is monetary authority's signal given by (A.6).

Minimizing the Volatility of Target Price Index. We first characterize the policy that minimizes the volatility of the target price index $\sum_{i=1}^n \psi_i^* \log p_i$.

By assumption, fraction ϕ_i of firms in industry i can set their prices flexibly, while the remaining $1-\phi_i$ fraction are subject to full nominal rigidities. As a result, the log nominal price of industry i satisfies $\log p_i = \phi_i(\alpha_i \log w - \log z_i + \sum_{j=1}^n a_{ij} \log p_j)$. Writing this equation in matrix form and solving for the vector of log nominal prices, we get

$$\log p = (\mathbf{I} - \mathbf{\Phi A})^{-1} \mathbf{\Phi}(\mathbf{I} - \mathbf{A})(\mathbf{1} \log w - \mathbf{L} \log z). \tag{B.47}$$

Next, recall that the vector of industry weights ψ^* satisfies (B.39). As a result,

$$\sum_{i=1}^n \psi_i^* \log p_i = v'(\mathbf{1} \log w - \mathbf{L} \log z),$$

where

$$v' = \lambda'\big[\mathrm{diag}(\theta)\big(\mathbf{I} - \mathrm{diag}(\mathbf{Q1})\big)(\mathbf{I} - \mathbf{A}) + \mathrm{diag}(\mathbf{Q1}) - \mathrm{diag}(\mathbf{AQ1})\mathbf{A}\big]\mathbf{Q} \tag{B.48}$$

and we are using the following identity: $\mathbf{L}(\mathbf{I} - \mathbf{\Phi})\mathbf{\Phi}^{-1}(\mathbf{I} - \mathbf{\Phi A})^{-1}\mathbf{\Phi}(\mathbf{I} - \mathbf{A}) = \mathbf{Q}$. Since $\log w = \kappa'\hat{\omega} = \kappa'(\log z + \hat{\epsilon})$, the volatility of the target price index is equal to

$$\mathrm{var}\left(\sum_{s=1}^n \psi_s^* \log p_s\right) = \delta^2 \sigma_z^2 \big\| v'\big(\mathbf{L} - \mathbf{1}\kappa'\big)\big\|_2^2 + \delta^2 \hat{\sigma}^2 \big\| v'\mathbf{1}\kappa'\big\|_2^2.$$

Optimizing the above with respect to κ implies that the policy that minimizes the volatility of the target price index is given by

$$\kappa' = \frac{\sigma_z^2}{\sigma_z^2 + \hat{\sigma}^2} \frac{v'\mathbf{L}}{v'\mathbf{1}}, \tag{B.49}$$

where v is given by (B.48).

Optimal Policy. We next characterize the optimal policy and show that it coincides with policy (B.49), which minimizes the volatility of the target price index.

By Proposition 6, welfare loss due to the presence of nominal rigidities is given by (36). Therefore, when $\gamma = 1$ and $\eta \to \infty$, the expected welfare loss is equal to

$$\mathbb{E}\big[W - W^*\big] = -\frac{1}{2}\lambda'\big(\mathrm{diag}(\theta)\mathbb{E}[\vartheta] + \mathrm{diag}\big(\mathbb{E}\big[\bar{e}\bar{e}'\big]\big) - \mathrm{diag}\big(\mathbf{A}\mathbb{E}\big[\bar{e}\bar{e}'\big]\mathbf{A}'\big)\big) + o\big(\delta^2\big), \tag{B.50}$$

where $\bar{e}$ and ϑ denote the vectors of cross-sectional mean and dispersion of pricing errors defined in (34) and (35), respectively.

To determine the right-hand side of (B.50), we make two observations. First, note that $\bar{e} = \mathbf{Q}(\mathbf{L} \log z - \mathbf{1} \log w)$, where $\mathbf{Q} = (\mathbf{I} - \mathbf{\Phi A})^{-1}(\mathbf{I} - \mathbf{\Phi})$. This follows from the fact that the vectors of log nominal prices in the sticky- and flexible-price equilibria are given by (B.47) and $\log p^* = \mathbf{1} \log w - \mathbf{L} \log z$, respectively. As a result,

$$\mathbb{E}\big[\bar{e}\bar{e}'\big] = \delta^2 \sigma_z^2 \mathbf{Q}\big(\mathbf{L} - \mathbf{1}\kappa'\big)\big(\mathbf{L}' - \kappa\mathbf{1}'\big)\mathbf{Q}' + \delta^2 \hat{\sigma}^2 \mathbf{Q1}\kappa'\kappa\mathbf{1}'\mathbf{Q}', \tag{B.51}$$

where we are using the fact that $\log w = \sum_{i=1}^n \kappa_i \hat{\omega}_i$. Second, since a fraction ϕ_i of firms in industry i can set their prices flexibly, while the remaining $1-\phi_i$ fraction face full nominal

TABLE C.I

EXPECTED WELFARE LOSS UNDER VARIOUS POLICIES.

(η, γ, ϕ_w)	Optimal Policy	Output-Gap Stabilization	Consumption Weighted	Domar Weighted	Stickiness-Adjusted CPI
$(0.5, 0.1, 0.25)$	0.64	0.65	1.42	1.30	1.26
$(0.5, 0.1, 0.30)$	0.64	0.65	1.18	1.09	1.03
$(0.5, 1, 0.25)$	0.64	0.64	1.25	1.17	1.13
$(0.5, 1, 0.30)$	0.63	0.64	1.08	1.02	0.97
$(0.5, 2, 0.25)$	0.63	0.64	1.15	1.10	1.05
$(0.5, 2, 0.30)$	0.63	0.64	1.03	0.97	0.93
$(1, 0.1, 0.25)$	0.65	0.65	1.79	1.63	1.53
$(1, 0.1, 0.30)$	0.64	0.66	1.41	1.27	1.18
$(1, 1, 0.25)$	0.64	0.65	1.37	1.27	1.22
$(1, 1, 0.30)$	0.64	0.65	1.16	1.07	1.02
$(1, 2, 0.25)$	0.63	0.64	1.20	1.14	1.09
$(1, 2, 0.30)$	0.63	0.64	1.06	1.00	0.96
$(2, 0.1, 0.25)$	0.66	0.67	2.52	2.21	2.03
$(2, 0.1, 0.30)$	0.65	0.67	1.85	1.59	1.46
$(2, 1, 0.25)$	0.64	0.65	1.49	1.37	1.30
$(2, 1, 0.30)$	0.64	0.65	1.23	1.13	1.07
$(2, 2, 0.25)$	0.64	0.64	1.24	1.17	1.12
$(2, 2, 0.30)$	0.63	0.64	1.08	1.02	0.97

Note: The table reports the expected welfare loss due to the presence of nominal rigidities under various monetary policies as a percentage of steady-state consumption. The expected welfare loss is calculated by simulating the exact model for 10,000 draws of the vector of productivity shocks. Parameters η, γ, and ϕ_w denote the Frisch elasticity of labor supply, the household's coefficient of relative risk aversion, and the degree of wage flexibility, respectively.

rigidities, price dispersion in industry i is equal to $\vartheta_i = (1/\phi_i - 1)\log^2 p_i$, where the log nominal price of industry i in the sticky-price equilibrium satisfies (B.47). As a result,

$$\mathbb{E}[\vartheta] = \delta^2\big(\mathbf{\Phi}^{-1} - \mathbf{I}\big)\operatorname{diag}\big((\mathbf{I}-\mathbf{Q})\big[\sigma_z^2\big(\mathbf{L}-\mathbf{1}\kappa'\big)\big(\mathbf{L}'-\kappa\mathbf{1}'\big)+\hat{\sigma}^2\mathbf{1}\kappa'\kappa\mathbf{1}'\big]\big(\mathbf{I}-\mathbf{Q}'\big)\big). \quad \text{(B.52)}$$

With the expressions in (B.51) and (B.52) in hand, we then minimize the expected welfare loss (B.50) with respect to κ, leading to the corresponding first-order condition: $\sigma_z^2 v'(\mathbf{1}\kappa' - \mathbf{L}) + \hat{\sigma}^2 v'\mathbf{1}\kappa' = 0$, where v is defined in (B.48). This, in turn implies that the policy that minimizes the expected welfare loss is given by $\kappa' = \frac{\sigma_z^2}{(\sigma_z^2+\hat{\sigma}^2)v'\mathbf{1}} v'\mathbf{L}$, which coincides with policy (B.49), which minimizes the volatility of target price index $\sum_{i=1}^n \psi_i^* \log p_i$.

APPENDIX C: ROBUSTNESS CHECKS

This Appendix contains some robustness checks for our quantitative analysis, where we report the expected welfare loss under the optimal policy and the four alternative policies considered in Section 5 for different parameter values. We calculate the expected welfare loss of each of the five policies (relative to the flexible-price equilibrium and measured as a fraction of steady-state consumption) by simulating the exact model for 10,000 draws of the vector of productivity shocks. We consider parameter values on a grid with $\eta \in \{0.5, 1, 2\}$, $\gamma \in \{0.1, 1, 2\}$, and $\phi_w \in \{0.25, 0.30\}$, where η, γ, and ϕ_w denote the Frisch elasticity of labor supply, the household's coefficient of relative risk aversion, and the degree of wage flexibility, respectively. Aside from the values discussed in the main body of the paper, we choose the remaining parameter values as follows. We consider $\eta = 0.5$

and $\gamma = 2$ as in McKay, Nakamura, and Steinsson (2016) and $\phi_w = 0.25$ as in Galí (2008). We also consider the case of $\eta = 1$ and $\gamma = 1$ as a natural benchmark.

Table C.I reports the results. Three observations are immediate. First, the expected welfare loss under the optimal policy is fairly robust to the choice of parameter values, taking a value between 0.63% and 0.66% of steady-state consumption. Second, irrespective of the calibration, the policy that stabilizes the output gap is nearly optimal: among all specifications, the wedge between the optimal and the output-gap-stabilization policies never exceeds 0.03 percentage points. Finally, while the expected welfare loss under the policy that targets the consumer price index is more sensitive to the choice of parameters, this policy significantly underperforms the optimal policy irrespective of the specification.

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Co-editor Chad Jones handled this manuscript.

Manuscript received 24 June, 2020; final version accepted 5 November, 2021; available online 30 November, 2021.