SUPPLEMENT TO "IDENTIFICATION AT THE ZERO LOWER BOUND"

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SOPHOCLES MAVROEIDIS
Department of Economics, University of Oxford

This document contains additional derivations, computational details, and additional simulation results to supplement the main paper. All equations in this supplement are prefixed by "S". Numbers without prefix refer to objects in the main paper.

APPENDIX C: A SIMPLE MODEL OF QE

THIS IS A SIMPLIFIED VERSION of the New Keynesian model of bond market segmentation that appeared in Ikeda, Li, Mavroeidis, and Zanetti (2020) and is based on Chen, Cúrdia, and Ferrero (2012). The economy consists of two types of households. A fraction ω_r of type "r" households can only trade long-term government bonds. The remaining $1-\omega_r$ households of type "u" can purchase both short-term and long-term government bonds, the latter subject to a trading cost ζ_t. This trading cost gives rise to a term premium, that is, a spread between long-term and short-term yields, that the central bank can manipulate by purchasing long-term bonds. The term premium affects aggregate demand through the consumption decisions of constrained households. This generates an unconventional monetary policy channel.

The transmission mechanism of monetary policy is obtained from the equilibrium conditions of households and firms in the economy. Households choose consumption to maximize an isoelastic utility function and firms set prices subject to Calvo frictions. These give rise to a Euler equation for output and a Phillips curve, respectively. Equation (1) in the paper can be derived by combining those two equations. I will derive the Euler equation in some detail in order to illustrate the origins of the QE channel. The Phillips curve derivation is standard and is therefore omitted.

Up to a log-linear approximation, the relevant first-order conditions of the households' optimization problem can be written as

$$0 = E_t\left[-\frac{1}{\sigma}\left(\hat{c}^u_{t+1}-\hat{c}^u_t\right)+\hat{r}_t-\pi_{t+1}\right], \tag{S.1}$$

$$\frac{\zeta}{1+\zeta}\hat{\zeta}_t = E_t\left[-\frac{1}{\sigma}\left(\hat{c}^u_{t+1}-\hat{c}^u_t\right)+\hat{R}_{L,t+1}-\pi_{t+1}\right], \tag{S.2}$$

$$0 = E_t\left[-\frac{1}{\sigma}\left(\hat{c}^r_{t+1}-\hat{c}^r_t\right)+\hat{R}_{L,t+1}-\pi_{t+1}\right], \tag{S.3}$$

where σ is the elasticity of intertemporal substitution, ζ is the steady-state value of ζ_t, hatted variables denote log-deviations from steady state, c^j_t is consumption of household $j\in\{u,r\}$, r_t is the short-term nominal interest rate, and $R_{L,t}$ is the gross yield on long-term government bonds from period $t-1$ to t.[1] Goods market clearing yields

$$\hat{y}_t = \omega_r\hat{c}^r_t+(1-\omega_r)\hat{c}^u_t, \tag{S.4}$$

Sophocles Mavroeidis: sophocles.mavroeidis@economics.ox.ac.uk

[1] I do not put a hat over π_t because I assume a zero inflation target for simplicity.

where y_t is output, and I have assumed, for simplicity, that in steady state $c^u = c^r$, which implies $c^u = c^r = y$. Multiplying (S.1) and (S.3) by $(1 - \omega_r)$ and ω_r, respectively, and adding them yields

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma E_t\big[(1-\omega_r)\hat{r}_t + \omega_r\hat{R}_{L,t+1} - \pi_{t+1}\big]. \tag{S.5}$$

Subtracting (S.1) from (S.2) yields

$$E_t(\hat{R}_{L,t+1}) = \hat{r}_t + \frac{\zeta}{1+\zeta}\hat{\zeta}_t, \tag{S.6}$$

which establishes that the term premium between long and short yields is proportional to $\hat{\zeta}_t$. Substituting for $E_t(\hat{R}_{L,t+1})$ in (S.5) using (S.6) yields

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma\left(\hat{r}_t + \omega_r\frac{\zeta}{1+\zeta}\hat{\zeta}_t\right) + \sigma E_t(\pi_{t+1}). \tag{S.7}$$

Next, assume that the cost of trading long-term bonds depends on their supply, $b_{L,t}$, that is,

$$\hat{\zeta}_t = \rho_\zeta\hat{b}_{L,t}, \quad \rho_\zeta \geq 0.$$

Substituting for $\hat{\zeta}_t$ in (S.7) yields the Euler equation

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma\left(\hat{r}_t + \omega_r\frac{\zeta}{1+\zeta}\rho_\zeta\hat{b}_{L,t}\right) + \sigma E_t(\pi_{t+1}). \tag{S.8}$$

The second equation is a standard New Keynesian Phillips curve that links inflation to output:

$$\pi_t = \delta E_t(\pi_{t+1}) + \varpi\hat{y}_t + \varepsilon_{1t}, \tag{S.9}$$

where δ is the average discount factor of the two households, $\varpi \geq 0$ is a parameter that depends on the degree of price stickiness (the Calvo parameter), and ε_{1t} is proportional to an i.i.d. technology shock. Substituting for $\hat{y}_t$ in (S.9) using (S.8) yields

$$\pi_t = (\delta + \sigma\varpi)E_t(\pi_{t+1}) + \varpi E_t(\hat{y}_{t+1}) - \varpi\sigma\left(\hat{r}_t + \omega_r\frac{\zeta}{1+\zeta}\rho_\zeta\hat{b}_{L,t}\right) + \varepsilon_{1t}. \tag{S.10}$$

Finally, at an equilibrium in which inflation and output depend only on the exogenous shocks $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$, which are the only state variables in the system, and when the shocks have no memory, $E_t(\pi_{t+1})$ and $E_t(\hat{y}_{t+1})$ will be equal to the corresponding unconditional expectations, which are constants.[2] Therefore, (S.10) reduces to equation (1) in the paper by setting $c = (\delta + \sigma\varpi)E(\pi_{t+1}) + \varpi E(\hat{y}_{t+1})$, $\beta = -\varpi\sigma$, $\hat{r}_t = r_t - r^n$, where r^n is the discount rate of the unconstrained households, and $\varphi = \omega_r\frac{\zeta}{1+\zeta}\rho_\zeta\beta$, and dropping the hat from $b_{L,t}$ for simplicity. The parameter φ depends on the fraction of constrained households, ω_r, and the sensitivity of the term premium to long-term asset holdings, $\frac{\zeta}{1+\zeta}\rho_\zeta$.

[2]Such an equilibrium always exists if the volatility of the shocks is not too large; see Mendes (2011).

APPENDIX D: FORWARD GUIDANCE RULES

Debortoli, Gali, and Gambetti (2019) discussed the following two inertial policy rules:

$$r_t = \max\{0, \phi_r r_{t-1} + (1-\phi_r)(\rho + \phi_\pi \pi_t + \phi_y \Delta y_t)\}, \tag{S.11}$$

and

$$r_t = \max(0, r_t^*), \tag{S.12a}$$

$$r_t^* = \phi_r r_{t-1}^* + (1-\phi_r)(\rho + \phi_\pi \pi_t + \phi_y \Delta y_t), \tag{S.12b}$$

where I have set the inflation target to zero, and Δy_t is output growth. Both of these rules are nested within equation (19) of the CKSVAR, with $Y_{1t} = (\pi_t, \Delta y_t)'$and $Y_{2t} = r_t$. Rule (S.11) sets the coefficients on $Y_{2,t-1}$ and $Y_{2,t-1}^*$ as $B_{22} = \phi_r$ and $B_{22}^* = 0$, respectively, while rule (S.12) sets them as $B_{22} = 0$ and $B_{22}^* = \phi_r$. Debortoli, Gali, and Gambetti (2019) argued rule (S.12) is consistent with forward guidance, because it will tend to keep interest rates at zero for longer than rule (S.11). It also ensures policy reaction is the same across regimes, and so it is consistent with the ZLB irrelevance hypothesis that the paper put forward.

Reifschneider and Williams (2000) proposed a slightly more elaborate policy rule for forward guidance:

$$r_t^* = r_t^{\text{Taylor}} - \alpha Z_t, \quad Z_t = Z_{t-1} + d_t, d_t := r_t - r_t^{\text{Taylor}}, \tag{S.13a}$$

$$r_t = \max(r_t^*, 0),$$

$$r_t^{\text{Taylor}} = \rho + \phi_\pi \pi_t + \phi_y y_t, \tag{S.13b}$$

where y_t is the output gap, and the inflation target is zero. Differencing (S.13a) yields

$$r_t^* = r_{t-1}^* + \Delta r_t^{\text{Taylor}} - \alpha(r_t - r_t^{\text{Taylor}}).$$

Substituting for r_t^{Taylor} using (S.13b) yields

$$\begin{aligned} r_t^* &= r_{t-1}^* + \phi_\pi \Delta \pi_t + \phi_y \Delta y_t - \alpha r_t + \alpha(\rho + \phi_\pi \pi_t + \phi_y y_t) \\ &= \alpha\rho - \alpha r_t + (1+\alpha)(\phi_\pi \pi_t + \phi_y y_t) - (\phi_\pi \pi_{t-1} + \phi_y y_{t-1}) + r_{t-1}^*. \end{aligned}$$

This is again nested within equation (19) of the CKSVAR with $Y_{1t} = (\pi_t, y_t)'$, $Y_{2t} = r_t$, $Y_{2t}^* = r_t^*$, $X_{1t} = (1, \pi_{t-1}, y_{t-1})'$, $X_{2t} = r_{t-1}$, $X_{2t}^* = r_{t-1}^*$, and parameters $A_{21} = -(1+\alpha)(\phi_\pi, \phi_y)$, $A_{22} = \alpha$, $A_{22}^* = 1$, $B_{21} = (\alpha\rho, -\phi_\pi, -\phi_y)$, $B_{22} = 0$ and $B_{22}^* = 1$.

More examples of forward guidance policy rules that are nested within the CKSVAR are discussed in Ikeda et al. (2020).

APPENDIX E: COMPUTATIONAL DETAILS

E.1. *Likelihood*

To compute the likelihood, we need to obtain the prediction error densities. The first step is to write the model in state-space form. Define

$$s_t = \begin{pmatrix} \mathbf{y}_t \\ \vdots \\ \mathbf{y}_{t-p+1} \end{pmatrix}, \qquad \underset{(k+1)\times 1}{\mathbf{y}_t} = \begin{pmatrix} Y_t \\ \overline{Y_{2t}^*} \end{pmatrix},$$

and write the state transition equation as

$$s_t = F(s_{t-1}, u_t; \psi) = \begin{pmatrix} F_1(s_{t-1}, u_t; \psi) \\ \mathbf{y}_{t-1} \\ \vdots \\ \mathbf{y}_{t-p+1} \end{pmatrix}, \tag{S.14}$$

$$F_1(s_{t-1}, u_t; \psi) = \begin{pmatrix} \overline{C}_1 X_t + \overline{C}_1^* \overline{X}_t^* + u_{1t} - \widetilde{\beta} D_t \big(\overline{C}_2 X_t + \overline{C}_2^* \overline{X}_t^* + u_{2t} - b\big) \\ \max\big(b, \overline{C}_2 X_t + \overline{C}_2^* \overline{X}_t^* + u_{2t}\big) \\ \overline{C}_2 X_t + \overline{C}_2^* \overline{X}_t^* + u_{2t} \end{pmatrix},$$

and the observation equation as

$$Y_t = \big(I_k \quad 0_{k\times 1+(p-1)(k+1)}\big)\, s_t. \tag{S.15}$$

Next, I will derive the predictive density and mass functions. With Gaussian errors, the joint predictive density of Y_t corresponding to the observations with $D_t = 0$ is

$$f_0(Y_t|s_{t-1}, \psi) = |\Omega|^{-1/2} \exp\Big\{-\frac{1}{2}\operatorname{tr}\big(\big(Y_t - \overline{C} X_t - \overline{C}^* \overline{X}_t^*\big) \times \big(Y_t - \overline{C} X_t - \overline{C}^* \overline{X}_t^*\big)' \Omega^{-1}\big)\Big\}. \tag{S.16}$$

At $D_t = 1$, the predictive density of Y_{1t} can be written as

$$f_1(Y_{1t}|s_{t-1}, \psi) := |\Xi_1|^{-\frac{1}{2}} \exp\Big[-\frac{1}{2}(Y_{1t} - \mu_{1t})' \Xi_1^{-1} (Y_{1t} - \mu_{1t})\Big], \tag{S.17}$$

$$\mu_{1t} := \widetilde{\beta} b + (\overline{C}_1 - \widetilde{\beta}\overline{C}_2) X_t + \big(\overline{C}_1^* - \widetilde{\beta}\overline{C}_2^*\big)\overline{X}_t^*, \tag{S.18}$$

$$\begin{aligned} \Xi_1 &:= \Omega_{1.2} + \widetilde{\delta}\widetilde{\delta}'\tau^2 = \big(I_{k-1} \quad -\widetilde{\beta}\big)\, \Omega \begin{pmatrix} I_{k-1} \\ -\widetilde{\beta}' \end{pmatrix}, \\ \widetilde{\delta} &= \Omega_{12}\omega_{22}^{-1} - \widetilde{\beta}, \end{aligned} \tag{S.19}$$

where $\Omega_{1.2} = \Omega_{11} - \Omega_{12}\omega_{22}^{-1}\Omega_{21}$, and $\tau = \sqrt{\omega_{22}}$. Next,

$$u_{2t}|Y_{1t}, s_{t-1} \sim N\big(\mu_{2t}, \tau_2^2\big) \quad \text{with} \tag{S.20}$$

$$\mu_{2t} := \tau^2 \widetilde{\delta}' \Xi_1^{-1} (Y_{1t} - \mu_{1t}),\ \tau_2 = \tau\sqrt{\big(1 - \tau^2 \widetilde{\delta}' \Xi_1^{-1} \widetilde{\delta}\big)}. \tag{S.21}$$

Hence,

$$\Pr(D_t = 1|Y_{1t}, s_{t-1}, \psi) = \Phi\Big(\frac{b - \overline{C}_2 X_t - \overline{C}_2^* \overline{X}_t^* - \mu_{2t}}{\tau_2}\Big). \tag{S.22}$$

In the case of the KSVAR model, there are no latent lags ($\overline{C}^* = 0$, $\overline{C} = C$), so the log-likelihood is available analytically:

$$\log L(\psi) = \sum_{t=1}^{T} (1 - D_t) \log f_0(Y_t|s_{t-1}, \psi)$$

$$+\sum_{t=1}^{T} D_t \log\left(f_1(Y_{1t}|s_{t-1},\psi)\Phi\left(\frac{b-C_2X_t-\mu_{2t}}{\tau_2}\right)\right), \tag{S.23}$$

where $f_0(Y_t|s_{t-1},\theta)$ and $f_1(Y_{1t}|s_{t-1},\theta)$ are given by (S.16) and (S.17), respectively, with $\overline{C}^* = 0$.

The likelihood for the unrestricted CKSVAR ($\overline{C}^* \neq 0$) can be computed approximately by simulation (particle filtering). I provide two different simulation algorithms. The first is a sequential importance sampler (SIS), proposed originally by Lee (1999) for the univariate dynamic Tobit model. It is extended here to the CKSVAR model. The second algorithm is a fully adapted particle filter (FAPF), which is a sequential importance resampling algorithm designed to address the sample degeneracy problem. It was proposed by Malik and Pitt (2011) and is a special case of the auxiliary particle filter developed by Pitt and Shephard (1999).

Both algorithms require sampling from the predictive density of $\overline{Y}^*_{2t}$ conditional on Y_{1t}, $D_t = 1$, and s_{t-1}. From (26) and (S.20), we see that this is a truncated Normal with original mean $\mu^*_{2t} = \overline{C}_2X_t + \overline{C}^*_2\overline{X}^*_t + \mu_{2t}$ and standard deviation τ_2, where μ_{2t}, τ_2 are given in (S.21), that is,

$$f_2\big(Y^*_{2t}|Y_{1t}, D_t = 1, s_{t-1},\psi\big) = TN\big(\mu^*_{2t},\tau_2,\overline{Y}^*_{2t} < b\big). \tag{S.24}$$

Draws from this truncated distribution can be obtained using, for instance, the procedure in Lee (1999). Let $\xi_t^{(j)} \sim U[0,1]$ be i.i.d. uniform random draws, $j = 1,\ldots,M$. Then, a draw from $\overline{Y}^*_{2t}|Y_{1t}, s_{t-1}, \overline{Y}^*_{2t} < b$ is given by

$$\overline{Y}^{*(j)}_{2t} = \mu^*_{2t} + \tau_2\Phi^{-1}\left[\xi_t^{(j)}\Phi\left(\frac{b-\mu^*_{2t}}{\tau_2}\right)\right]. \tag{S.25}$$

ALGORITHM 1—SIS: *Sequential Importance Sampler*

1. Initialization. *For* $j = 1:M$, *set* $W_0^j = 1$ *and* $s_0^j = (\mathbf{y}_0^j,\ldots,\mathbf{y}_{-p+1}^j)$, *with* $\mathbf{y}^j_{-s} = (Y_0', Y_{2,0})'$, *for* $s = 0,\ldots,p-1$. (*In other words, initialize* $\overline{Y}^*_{2,-s}$ *at the observed values of* $Y_{2,-s}$.)
2. Recursion. *For* $t = 1:T$:
   (a) *For* $j = 1:M$, *compute the incremental weights*

$$w^j_{t-1|t} = p\big(Y_t|s^j_{t-1},\psi\big) = \begin{cases} f_0\big(Y_t|s^j_{t-1},\psi\big) & \text{if } D_t = 0,\\ f_1\big(Y_{1t}|s^j_{t-1},\psi\big)\Pr\big(D_t = 1|Y_{1t},s^j_{t-1},\psi\big) & \text{if } D_t = 1,\end{cases}$$

   where f_0, f_1, *and* $\Pr(D_t = 1|Y_{1t}, s_{t-1};\psi)$ *are given by* (S.16), (S.17), *and* (S.22), *resp., and*

$$S_t = \frac{1}{M}\sum_{j=1}^{M} w^j_{t-1|t}W^j_{t-1}.$$

   (b) *Sample* s_t^j *randomly from* $p(s_t|s^j_{t-1},Y_t)$. *That is,* $s_t^j = (\mathbf{y}_t^j,\mathbf{y}^j_{t-1},\ldots,\mathbf{y}^j_{t-p})$, *where* $\mathbf{y}_t^j = (Y_t', \overline{Y}^{*(j)}_{2t})$ *and* $\overline{Y}^{*(j)}_{2t}$ *is a draw from* $f_2(Y^*_{2t}|Y_{1t}, D_t = 1, s^j_{t-1},\psi)$ *using* (S.25).

(c) *Update the weights*:

$$W_t^j = \frac{w_{t-1|t}^j W_{t-1}^j}{S_t}.$$

3. *Likelihood approximation*:

$$\log \widehat{p}(Y_T|\psi) = \sum_{t=1}^{T} \log S_t.$$

If the draws $\xi_t^{(j)}$ are kept fixed across different values of ψ, the simulated likelihood in step 3 is smooth. Note that when $k = 1$ and $Y_t = Y_{2t}$ (no Y_{1t} variables), the model reduces to a univariate dynamic Tobit model, and Algorithm 1 reduces exactly to the sequential importance sampler proposed by Lee (1999). A possible weakness of this algorithm is sample degeneracy, which arises when all but a few weights W_t^J are zero. To gauge possible sample degeneracy, we can look at the effective sample size (ESS), as recommended by Herbst and Schorfheide (2015):

$$\text{ESS}_t = \frac{M}{\frac{1}{M}\sum_{j=1}^{M}\left(W_t^j\right)^2}. \tag{S.26}$$

Next, I turn to the FAPF algorithm.

ALGORITHM 2—FAPF: *Fully Adapted Particle Filter*

1. Initialization. *For* $j = 1 : M$, *set* $s_0^j = (\mathbf{y}_0^j, \ldots, \mathbf{y}_{-p+1}^j)$, *with* $\mathbf{y}_{-s}^j = (Y_0', Y_{2,0})'$, *for* $s = 0, \ldots, p-1$. (*In other words, initialize* $\overline{Y}_{2,-s}^*$ *at the observed values of* $Y_{2,-s}$.)
2. Recursion. *For* $t = 1 : T$:
   (a) *For* $j = 1 : M$, *compute*

$$w_{t-1|t}^j = p\left(Y_t|s_{t-1}^j, \psi\right) = \begin{cases} f_0\left(Y_t|s_{t-1}^j, \psi\right) & \text{if } D_t = 0, \\ f_1\left(Y_{1t}|s_{t-1}^j, \psi\right)\Pr\left(D_t = 1|Y_{1t}, s_{t-1}^j, \psi\right) & \text{if } D_t = 1, \end{cases}$$

where f_0, f_1, *and* $\Pr(D_t = 1|Y_{1t}, s_{t-1}; \psi)$ *are given by* (S.16), (S.17), *and* (S.22), *resp., and*

$$\pi_{t-1|t}^j = \frac{w_{t-1|t}^j}{\sum_{j=1}^{M} w_{t-1|t}^j}.$$

   (b) *For* $j = 1 : M$, *sample* k_j *randomly from the multinomial distribution* $\{j, \pi_{t-1|t}^j\}$. *Then, set* $\tilde{s}_{t-1}^j = s_{t-1}^{k_j}$. (*This applies only to the elements in* s_{t-1}^j *that correspond to* X_t^{*j}, *since all the other elements are observed and constant across all* j. *That is,* $\tilde{s}_{t-1}^j = (\tilde{\mathbf{y}}_{t-1}^j, \ldots, \tilde{\mathbf{y}}_{t-p}^j)$, $\tilde{\mathbf{y}}_{t-s}^j = (Y_{t-1}', \overline{Y}_{2,t-s}^{*(k_j)})$, $s = 1, \ldots, p$.)

(c) *For* $j = 1 : M$, *sample* s_t^j *randomly from* $p(s_t|\tilde{s}_{t-1}^j, Y_t)$. *That is,* $s_t^j = (\mathbf{y}_t^j, \ldots, \tilde{\mathbf{y}}_{t-p}^j)$, *where* $\mathbf{y}_t^j = (Y_t', \overline{Y}_{2t}^{*(j)})$ *and* $\overline{Y}_{2t}^{*(j)}$ *is a draw from* $f_2(Y_{2t}^*|Y_{1t}, D_t = 1, \tilde{s}_{t-1}^j, \psi)$ *using* (S.25).

3. *Likelihood approximation*:

$$\ln \hat{p}(Y_T|\psi) = \sum_{t=1}^{T} \ln\left(\frac{1}{M}\sum_{j=1}^{M} w_{t-1|t}^j\right).$$

Many of the generic particle filtering algorithms used in the macro literature, described in Herbst and Schorfheide (2015), are inapplicable in a censoring context because of the absence of measurement error in the observation equation. It is, of course, possible to introduce a small measurement error in Y_{2t}, so that the constraint $Y_{2t} \geq b$ is not fully respected, but there is no reason to expect other particle filters discussed in Herbst and Schorfheide (2015) to estimate the likelihood more accurately than the FAPF algorithm described above.

Moments or quantiles of the filtering or smoothing distribution of any function $h(\cdot)$ of the latent states s_t can be computed using the drawn sample of particles. When we use Algorithm 2, simple average or quantiles of $h(s_t^j)$ produce the requisite average or quantiles of $h(s_t)$ conditional on $Y_1, \ldots, Y_t$ (the filtering density). For particles generated using Algorithm 1, we need to take weighted averages using the importance sampling weights W_t. Smoothing estimates of $h(s_t^j)$ can be obtained using weights W_T.

E.2. *Computation of the Identified Set*

Substitute for $\overline{\gamma}$ in (39) using Proposition 3 to get

$$\widetilde{\beta} = (1-\xi)\big(I - \xi\overline{\beta}\big(\Omega_{12}' - \Omega_{22}\overline{\beta}'\big)\big(\Omega_{11} - \Omega_{12}\overline{\beta}'\big)^{-1}\big)^{-1}\overline{\beta}. \tag{S.27}$$

For each value of $\xi \in [0, 1)$, the above equation defines a correspondence from $\Re^{k-1}$ to $\Re^{k-1}$. The range of $\overline{\beta}$ can then be obtained numerically by solving (S.27) for $\overline{\beta}$ as a function of the reduced-form parameters and ξ for each value of ξ, and gathering all the solutions in the set.

Rearranging (S.27) yields

$$\widetilde{\beta} = \xi\overline{\beta}\big(\Omega_{12}' - \Omega_{22}\overline{\beta}'\big)\big(\Omega_{11} - \Omega_{12}\overline{\beta}'\big)^{-1}\widetilde{\beta} + (1-\xi)\overline{\beta}. \tag{S.28}$$

Note that

$$\big(\Omega_{11} - \Omega_{12}\overline{\beta}'\big)^{-1} = \Omega_{11}^{-1} + \Omega_{11}^{-1}\Omega_{12}\big(1 - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}\big)^{-1}\overline{\beta}'\Omega_{11}^{-1}.$$

Hence,

$$\begin{aligned}
&\big(\Omega_{12}' - \Omega_{22}\overline{\beta}'\big)\big(\Omega_{11} - \Omega_{12}\overline{\beta}'\big)^{-1} \\
&\quad = \big(\Omega_{12}' - \Omega_{22}\overline{\beta}'\big)\Omega_{11}^{-1} + \frac{\big(\Omega_{12}'\Omega_{11}^{-1}\Omega_{12} - \Omega_{22}\overline{\beta}'\Omega_{11}^{-1}\Omega_{12}\big)\overline{\beta}'\Omega_{11}^{-1}}{1 - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}} \\
&\quad = \frac{\big(\Omega_{12}' - \Omega_{22}\overline{\beta}'\big)\Omega_{11}^{-1} + \Omega_{12}'\Omega_{11}^{-1}\big(\Omega_{12}\overline{\beta}'\Omega_{11}^{-1} - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}I_{k-1}\big)}{1 - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}}.
\end{aligned}$$

Substituting this back into (S.28), we get

$$\widetilde{\beta} = \xi\overline{\beta}\frac{(\Omega'_{12} - \Omega_{22}\overline{\beta}')\Omega_{11}^{-1} + \Omega'_{12}\Omega_{11}^{-1}(\Omega_{12}\overline{\beta}'\Omega_{11}^{-1} - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}I_{k-1})}{1 - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}}\widetilde{\beta} + (1-\xi)\overline{\beta}.$$

Multiplying both sides by $1 - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}$ yields

$$\begin{aligned}\widetilde{\beta} - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}\widetilde{\beta} = {} & \xi\overline{\beta}\Omega'_{12}\widetilde{\beta} - \xi\overline{\beta}\Omega_{22}\overline{\beta}'\Omega_{11}^{-1}\widetilde{\beta} + \xi\overline{\beta}\Omega'_{12}\Omega_{11}^{-1}\Omega_{12}\overline{\beta}'\Omega_{11}^{-1}\widetilde{\beta} \\ & - \xi\overline{\beta\beta}'\Omega_{11}^{-1}\Omega_{12}\Omega'_{12}\Omega_{11}^{-1}\widetilde{\beta} + (1-\xi)\overline{\beta}(1 - \overline{\beta}'\Omega_{11}^{-1}\Omega_{12}).\end{aligned}$$

Rearranging, we have

$$\begin{aligned}\widetilde{\beta} = {} & \widetilde{\beta}\Omega'_{12}\Omega_{11}^{-1}\overline{\beta} + \xi\Omega'_{12}\widetilde{\beta}\overline{\beta} + (1-\xi)\overline{\beta} - (1-\xi)\overline{\beta\beta}'\Omega_{11}^{-1}\Omega_{12} \\ & + \overline{\beta\beta}'\Omega_{11}^{-1}\widetilde{\beta}\xi\Omega'_{12}\Omega_{11}^{-1}\Omega_{12} - \overline{\beta\beta}'\Omega_{11}^{-1}\widetilde{\beta}\xi\Omega_{22} - \overline{\beta\beta}'\Omega_{11}^{-1}\Omega_{12}\Omega'_{12}\Omega_{11}^{-1}\widetilde{\beta}\xi \\ = {} & \big(\widetilde{\beta}\Omega'_{12}\Omega_{11}^{-1} + (\xi\Omega'_{12}\widetilde{\beta} + 1 - \xi)I_{k-1}\big)\overline{\beta} \\ & + \overline{\beta\beta}'\Omega_{11}^{-1}\big(\big((\Omega'_{12}\Omega_{11}^{-1}\Omega_{12} - \Omega_{22})I_{k-1} - \Omega_{12}\Omega'_{12}\Omega_{11}^{-1}\big)\widetilde{\beta}\xi - (1-\xi)\Omega_{12}\big).\end{aligned}$$

This can be written as

$$\widetilde{\beta} - \tilde{A}\overline{\beta} + \overline{\beta\beta}'\tilde{b} = 0, \tag{S.29}$$

where

$$\begin{aligned}\tilde{b} &:= -\Omega_{11}^{-1}\big(\big((\Omega'_{12}\Omega_{11}^{-1}\Omega_{12} - \Omega_{22})I_{k-1} - \Omega_{12}\Omega'_{12}\Omega_{11}^{-1}\big)\widetilde{\beta}\xi - (1-\xi)\Omega_{12}\big) \quad \text{and} \\ \tilde{A} &:= \widetilde{\beta}\Omega'_{12}\Omega_{11}^{-1} + (\xi\Omega'_{12}\widetilde{\beta} + 1 - \xi)I_{k-1}.\end{aligned}$$

Define

$$z := \tilde{b}'x \quad \text{and} \quad w := \tilde{b}'_{\perp}x,$$

where $\tilde{b}'_{\perp}\tilde{b}_{\perp} = 1$ and $\tilde{b}'_{\perp}\tilde{b} = 0$. Hence, rewrite (S.29) as

$$\widetilde{\beta} - \tilde{A}\tilde{b}(\tilde{b}'\tilde{b})^{-1}z - \tilde{A}\tilde{b}_{\perp}w + \tilde{b}(\tilde{b}'\tilde{b})^{-1}z^2 + \tilde{b}_{\perp}wz = 0.$$

Premultiply by $\tilde{b}'_{\perp}$ to get

$$\tilde{b}'_{\perp}\widetilde{\beta} - \tilde{b}'_{\perp}\tilde{A}\tilde{b}(\tilde{b}'\tilde{b})^{-1}z - \tilde{b}'_{\perp}\tilde{A}\tilde{b}_{\perp}w + wz = 0.$$

Solve that for w to get

$$w = (\tilde{b}'_{\perp}\tilde{A}\tilde{b}_{\perp} - z)^{-1}(\tilde{b}'_{\perp}\widetilde{\beta} - \tilde{b}'_{\perp}\tilde{A}\tilde{b}(\tilde{b}'\tilde{b})^{-1}z) = C_0(z)^{-1}c_1(z),$$

with

$$\begin{aligned}C_0(z) &:= (\tilde{b}'_{\perp}\tilde{A}\tilde{b}_{\perp} - z) \quad \text{and} \\ c_1(z) &:= \tilde{b}'_{\perp}\widetilde{\beta} - \tilde{b}'_{\perp}\tilde{A}\tilde{b}(\tilde{b}'\tilde{b})^{-1}z,\end{aligned}$$

provided that $\det(C_0(z)) \neq 0$.

Next, premultiply (S.29) by $\tilde{b}'$ and substitute for w to get

$$\tilde{b}'\widetilde{\beta} - \tilde{b}'\tilde{A}\tilde{b}\big(\tilde{b}'\tilde{b}\big)^{-1}z - \tilde{b}'\tilde{A}\tilde{b}_{\perp}C_0(z)^{-1}c_1(z) + z^2 = 0. \tag{S.30}$$

Now, notice that $C_0(z)^{-1} = C_0(z)^{\text{adj}}/\det(C_0(z))$, where C^{adj} is the adjoint of a square matrix C. Moreover, since $C_0(z)$ is of dimension $k-2$ and its elements are linear in z, $\det(C_0(z))$ is a polynomial in z of order at most $k-2$, and the elements of $C_0(z)^{\text{adj}}$ are polynomials in z of order at most $k-3$. For $k=2$, w is empty, so (S.29) is simply a quadratic in z. When $k>2$, $\det(C_0(z))$ is nonzero and we can multiply (S.30) by it to get

$$\begin{aligned} 0 = {} & \tilde{b}'\widetilde{\beta}\det\big(C_0(z)\big) + \tilde{b}'\tilde{A}\tilde{b}\big(\tilde{b}'\tilde{b}\big)^{-1}z\det\big(C_0(z)\big) \\ & - \tilde{b}'\tilde{A}\tilde{b}_{\perp}C_0(z)^{\text{adj}}c_1(z) + \det\big(C_0(z)\big)z^2. \end{aligned} \tag{S.31}$$

This is a polynomial equation of order k and has at most k solutions, denoted z_i, say. Then, the solutions for $\overline{\beta}$ are given by

$$\begin{aligned} \overline{\beta}_i &= \big[\tilde{b}\big(\tilde{b}'\tilde{b}\big)^{-1}, \tilde{b}_{\perp}\big]\begin{pmatrix} z_i \\ C_0(z_i)^{-1}c_1(z_i)\end{pmatrix} \\ &= \tilde{b}\big(\tilde{b}'\tilde{b}\big)^{-1}z_i + \tilde{b}_{\perp}C_0(z_i)^{-1}c_1(z_i). \end{aligned} \tag{S.32}$$

Below, I give some special cases.

Case $k=2$: In this case, w is empty, $\overline{\beta}$ is a scalar, and the equation (S.29) is a quadratic

$$\widetilde{\beta} - \tilde{A}\overline{\beta} + \overline{\beta}^2\tilde{b} = 0.$$

If $\tilde{A}^2 - 4\widetilde{\beta}\tilde{b} > 0$, the two real solutions are $\overline{\beta}_{1,2} = \frac{\tilde{A}\pm\sqrt{\tilde{A}^2-4\widetilde{\beta}\tilde{b}}}{2\tilde{b}}$.

Case $k=3$: In this case, w is a scalar, and the equation (S.31) can be written as a cubic in z, that is,

$$C_0(z)\tilde{b}'\widetilde{\beta} - C_0(z)\tilde{b}'\tilde{A}\tilde{b}\big(\tilde{b}'\tilde{b}\big)^{-1}z - \tilde{b}'\tilde{A}\tilde{b}_{\perp}c_1(z) + C_0(z)z^2 = 0, \tag{S.33}$$

since $C_0(z)$ is a scalar linear function of z. It can be shown that one of the roots of (S.33) satisfies $\Omega_{12}'\Omega_{11}^{-1}\overline{\beta} = 1$, which implies $\det(\Omega_{11} - \Omega_{12}\overline{\beta}') = 0$, and hence violates the equation for $\overline{\gamma} = (\Omega_{12}' - \Omega_{22}\overline{\beta}')(\Omega_{11} - \Omega_{12}\overline{\beta}')^{-1}$, so it is not a valid solution. The root in question is

$$z_1 = \frac{\Omega_{12}'\Omega_{11}^{-1}\big(\tilde{b}_{\perp}\tilde{b}'\tilde{A}\tilde{b} - \tilde{b}\tilde{b}'\tilde{A}\tilde{b}_{\perp}\big)}{\Omega_{12}'\Omega_{11}^{-1}\tilde{b}_{\perp}\big(\tilde{b}'\tilde{b}\big)}.$$

We can then factor out a term $z - z_1$ from (S.33), and obtain the remaining two roots from a quadratic equation. Therefore, there will be zero or two solutions for $\overline{\beta}$, as in the case $k=2$.

An algorithm for obtaining the identified set of the IRF (40) is as follows.

ALGORITHM 3—ID set: *Discretize the space* $(0,1)$ *into* R *equidistant points. For each* $r = 1:R$, *set* $\xi_r = \frac{r}{R+1}$ *and solve equation* (S.30).

1. *If no solution exists, proceed to the next* r.
2. *If* $0 < q_r \leq k$ *solutions exist, denote them* $z_{i,r}$, *and, for each* $i = 1:q_r$,
   (a) *derive* $\overline{\beta}_{i,r}$ *from* (S.32), $\overline{\gamma}_{i,r} = (\Omega_{12}' - \Omega_{22}\overline{\beta}_{i,r}')(\Omega_{11} - \Omega_{12}\overline{\beta}_{i,r}')^{-1}$, $\overline{A}_{22,i,r}^{-1} = \sqrt{(-\overline{\gamma}_{i,r}, 1)\Omega(-\overline{\gamma}_{i,r}, 1)'}$, *and* $\Xi_{1,i,r} = (I_{k-1}, -\overline{\beta}_{i,r})\Omega(I_{k-1}, -\overline{\beta}_{i,r})'$;
   (b) *for* $j = 1:M$,
      i. *draw independently* $\bar{\varepsilon}_{1t,i,r}^{j} \sim N(0, \Xi_{1,i,r})$ *and* $u_{t+h}^{j} \sim N(0,\Omega)$ *for* $h = 1,\ldots,H$;
      ii. *for any scalar* ς, *set*

$$u_{1t,i,r}^{j}(\varsigma) = (I_{k-1} - \overline{\beta}_{i,r}\overline{\gamma}_{i,r})^{-1}\big(\bar{\varepsilon}_{1t,i,r}^{j} - \overline{\beta}_{i,r}\varsigma\big),$$
$$u_{2t,i,r}^{j}(\varsigma) = (1 - \overline{\gamma}_{i,r}\overline{\beta}_{i,r})^{-1}\big(\varsigma - \overline{\gamma}_{i,r}\bar{\varepsilon}_{1t,i,r}^{j}\big),$$

   and compute $Y_{t,i,r}^{j}(\varsigma)$ *using* (24)–(25) *with* $u_{t,i,r}^{j}(\varsigma)$ *in place of* u_t, *and iterate forward to obtain* $Y_{t+h,i,r}^{j}(\varsigma)$ *using* u_{t+h}^{j} *computed in step i. Set* $\varsigma = 1$ *for a one-unit* (*e.g.*, 100 *basis points*) *impulse to the policy shock* $\overline{\varepsilon}_{2t}$, *or* $\varsigma = \overline{A}_{22,i,r}^{-1}$ *for a one-standard deviation impulse*;
   (c) *compute*

$$\widehat{\mathrm{IRF}}_{h,t,i,r}(\varsigma) = \frac{1}{M}\sum_{j=1}^{M}\big(Y_{t+h,i,r}^{j}(\varsigma) - Y_{t+h,i,r}^{j}(0)\big).$$

The identified set is given by the collection of $\widehat{\mathrm{IRF}}_{h,t,i,r}(\varsigma)$ *over* $i = 1:q_r$, $r = 1:R$, *and the single point-identified IRF at* $\xi = 0$.

E.3. *IRFs and Local Projections*

I will briefly discuss the difficulty in getting a local projection-like representation of the IRF in a dynamic Tobit model, which is a univariate CKSVAR(1). The model is given by the equations

$$\begin{aligned} y_t^* &= \rho y_{t-1} + \rho^* \min(y_{t-1} - b, 0) + u_t \\ &= \rho y_{t-1} + \rho^* D_{t-1}\big(y_{t-1}^* - b\big) + u_t, \quad D_t = 1\big\{y_t^* < b\big\}, \\ y_t &= \max\big(y_t^*, b\big) = (1 - D_t)y_t^*. \end{aligned}$$

Hence,

$$E_t(y_{t+1}) = \big(\rho y_t + \rho^* D_t\big(y_t^* - b\big)\big)\left(1 - \Phi\left(\frac{b - \rho y_t - \rho^* D_t\big(y_t^* - b\big)}{\sigma}\right)\right) + \sigma\phi\left(\frac{b - \rho y_t - \rho^* D_t\big(y_t^* - b\big)}{\sigma}\right).$$

In a linear model ($\rho^* = 0,\ b = -\infty$), the one-period-ahead impulse response is ρ, which coincides with the coefficient on y_t in the local projection $E_t(y_{t+1}) = \rho y_t$. In that case, the

coefficient ρ corresponds to both $\frac{\partial E_t(y_{t+1})}{\partial u_t} = \frac{\partial E_t(y_{t+1})}{\partial y_t}$ and $E(y_{t+1}|u_t = 1, y_{t-1}) - E(y_{t+1}|u_t = 0, y_{t-1})$. None of these properties hold in the dynamic Tobit model. For example, if we go with $\frac{\partial E_t(y_{t+1})}{\partial u_t}$ as our definition of the impulse response, we will not be able to obtain it from the slope of the conditional expectation function $E_t(y_{t+1})$ with respect to y_t. One problem is that the function $E_t(y_{t+1})$ is non-differentiable at $u_t = b - \rho y_{t-1} + \rho^* D_{t-1}(y^*_{t-1} - b)$, that is, exactly at the boundary. Another problem is that we still need to rely on the parametric structure of the model to uncover the impulse response from $E_t(y_{t+1})$. For example, we need to compute $\frac{\partial E_t(y_{t+1})}{\partial u_t}$, which at all points $y^*_t \neq b$ is given by

$$\frac{\partial E_t(y_{t+1})}{\partial u_t} = \begin{cases} \rho\left(1 - \Phi\left(\frac{b - \rho y_t}{\sigma}\right) + \frac{b}{\sigma}\phi\left(\frac{b - \rho y_t}{\sigma}\right)\right) \\ \quad \text{if } D_t = 0, \\ \rho^*\left(1 - \Phi\left(\frac{b - \rho b - \rho^*(y^*_t - b)}{\sigma}\right) + \frac{b}{\sigma}\phi\left(\frac{b - \rho b - \rho^*(y^*_t - b)}{\sigma}\right)\right) \\ \quad \text{if } D_t = 1. \end{cases}$$

There is no clear way to obtain the above impulse response from a local projection of y_{t+1} on simple nonlinear transformations of y_t, such as powers or interactions with the regime indicator.

APPENDIX F: ADDITIONAL SIMULATION RESULTS

Tables F.I, F.II, and F.III report the bias, standard deviation, and root mean square error of the ML estimator of the coefficients of the CKSVAR, KSVAR, and CSVAR models, respectively, under the data generating process DGP1 described in Appendix B of the paper. The description of the parameter names in the tables is given in Table B.I of the paper. Recall that under this DGP, all three models are correctly specified, since the coefficients on both lagged and observed Y_2 are zero in all equations, and there is no kink, that is, the true $\widetilde{\beta} = 0$. The CKSVAR and CSVAR likelihoods are computed using the SIS algorithm with $R = 1000$ particles. The sample size varies as $T \in \{100, 250, 1000\}$. We notice that the bias of the MLE is negligible for all the parameters in all cases, and that the standard deviation and RMSE fall at a rate $\sqrt{T}$, in accordance with asymptotic theory. The results for DGP2 and DGP3 are very similar and are therefore omitted.

F.1. *Alternative DGP*

The DGPs in the previous simulations have the property that the frequency of the ZLB regime is around 50%. I reran those simulations with a slight modification to the DGPs to match the frequency in the sample of the empirical application in the paper. Specifically, I reduce the lower bound b to a level that makes the frequency of the ZLB regime equal to 11%. The results are given in Figure F.1 and Table F.IV. The results are very similar to the ones reported in Figure B.1 and Table B.II in the Appendix of the main paper: the Normal approximation of the sampling distribution of the MLE appears to be very good, and the bias is negligible.

TABLE F.I

BIAS, STANDARD DEVIATION, AND ROOT MEAN SQUARE ERROR OF MAXIMUM LIKELIHOOD ESTIMATOR OF PARAMETERS OF CKSVAR(1) MODEL.[a]

ML-CKSVAR	$T = 100$			$T = 250$			$T = 1000$		
Parameter	Bias	sd	RMSE	Bias	sd	RMSE	Bias	sd	RMSE
τ	−0.048	0.111	0.121	−0.017	0.068	0.070	−0.003	0.035	0.035
Eq.3 Constant	0.001	0.293	0.293	−0.006	0.175	0.176	0.004	0.085	0.085
Eq.3 Y11_1	−0.001	0.111	0.111	0.000	0.061	0.061	0.000	0.032	0.032
Eq.3 Y12_1	−0.004	0.109	0.109	−0.000	0.064	0.064	−0.000	0.031	0.031
Eq.3 Y2_1	−0.031	0.289	0.291	−0.010	0.173	0.173	−0.003	0.082	0.082
Eq.3 lY2_1	−0.022	0.435	0.436	−0.013	0.258	0.258	0.003	0.122	0.122
$\tilde{\beta}_1$	0.012	0.586	0.586	−0.008	0.356	0.356	0.000	0.176	0.176
$\tilde{\beta}_2$	−0.011	0.606	0.606	−0.004	0.359	0.359	−0.003	0.168	0.168
Eq.1 Constant	0.000	0.360	0.360	0.002	0.213	0.213	0.007	0.105	0.105
Eq.1 Y11_1	−0.034	0.098	0.104	−0.011	0.057	0.058	−0.002	0.029	0.029
Eq.1 Y12_1	−0.004	0.108	0.108	0.002	0.060	0.060	0.001	0.028	0.028
Eq.1 Y2_1	−0.004	0.281	0.281	−0.002	0.163	0.163	−0.006	0.079	0.079
Eq.2 Constant	0.006	0.356	0.356	0.005	0.214	0.214	0.002	0.102	0.103
Eq.2 Y11_1	0.004	0.103	0.103	0.000	0.058	0.058	0.000	0.028	0.028
Eq.2 Y12_1	−0.029	0.103	0.107	−0.009	0.057	0.058	−0.002	0.029	0.029
Eq.2 Y2_1	0.000	0.269	0.269	−0.001	0.159	0.159	0.001	0.078	0.078
Eq.1 lY2_1	0.009	0.403	0.403	0.005	0.232	0.232	0.010	0.112	0.113
Eq.2 lY2_1	−0.003	0.408	0.408	0.002	0.233	0.233	0.004	0.115	0.115
δ_1	0.005	0.260	0.260	−0.001	0.157	0.157	0.001	0.075	0.075
δ_2	−0.005	0.260	0.260	−0.004	0.155	0.155	−0.001	0.073	0.073
Ch_11	−0.067	0.073	0.099	−0.025	0.044	0.051	−0.006	0.024	0.024
Ch_21	−0.005	0.111	0.111	−0.001	0.066	0.066	−0.000	0.031	0.031
Ch_22	−0.075	0.073	0.105	−0.028	0.046	0.054	−0.008	0.023	0.024

[a]Computed under DGP1 with $R = 1000$ particles using 1000 MC replications. Parameter names described in Table B.I.

TABLE F.II

BIAS, STANDARD DEVIATION, AND ROOT MEAN SQUARE ERROR OF MAXIMUM LIKELIHOOD ESTIMATOR OF PARAMETERS OF KSVAR(1) MODEL.[a]

ML-KSVAR	$T = 100$			$T = 250$			$T = 1000$		
Parameter	Bias	sd	RMSE	Bias	sd	RMSE	Bias	sd	RMSE
τ	−0.024	0.111	0.113	−0.008	0.068	0.069	−0.001	0.035	0.035
Eq.3 Constant	0.011	0.145	0.145	0.001	0.092	0.092	0.003	0.046	0.046
Eq.3 Y11_1	−0.001	0.103	0.103	0.001	0.060	0.060	−0.000	0.031	0.031
Eq.3 Y12_1	−0.004	0.102	0.102	−0.000	0.062	0.062	−0.000	0.030	0.030
Eq.3 Y2_1	−0.048	0.199	0.204	−0.019	0.122	0.124	−0.003	0.060	0.060
$\tilde{\beta}_1$	−0.003	0.571	0.571	−0.013	0.349	0.349	−0.001	0.174	0.174
$\tilde{\beta}_2$	−0.003	0.584	0.584	−0.001	0.348	0.348	−0.004	0.168	0.168
Eq.1 Constant	0.002	0.264	0.264	0.001	0.165	0.165	0.001	0.080	0.080
Eq.1 Y11_1	−0.033	0.093	0.099	−0.012	0.056	0.057	−0.002	0.028	0.028
Eq.1 Y12_1	−0.002	0.100	0.100	0.002	0.058	0.058	0.001	0.027	0.027
Eq.1 Y2_1	−0.002	0.197	0.197	−0.000	0.117	0.117	−0.001	0.057	0.057
Eq.2 Constant	0.004	0.258	0.258	0.003	0.158	0.158	0.001	0.078	0.078
Eq.2 Y11_1	0.006	0.096	0.096	0.001	0.057	0.057	0.000	0.027	0.027
Eq.2 Y12_1	−0.028	0.094	0.098	−0.008	0.055	0.056	−0.002	0.028	0.029
Eq.2 Y2_1	−0.001	0.189	0.189	−0.000	0.113	0.113	0.003	0.054	0.055
δ_1	−0.000	0.252	0.252	−0.003	0.156	0.156	0.001	0.075	0.075
δ_2	−0.003	0.253	0.253	−0.003	0.152	0.152	−0.001	0.073	0.073
Ch_11	−0.048	0.070	0.085	−0.018	0.044	0.047	−0.005	0.023	0.024
Ch_21	−0.003	0.108	0.108	−0.000	0.065	0.065	−0.000	0.031	0.031
Ch_22	−0.054	0.070	0.088	−0.020	0.045	0.050	−0.007	0.023	0.024

[a]Computed under DGP1 with $R = 1000$ particles using 1000 MC replications. Parameter names described in Table B.I.

TABLE F.III

BIAS, STANDARD DEVIATION, AND ROOT MEAN SQUARE ERROR OF MAXIMUM LIKELIHOOD ESTIMATOR OF PARAMETERS OF CSVAR(1) MODEL.[a]

ML-CSVAR	$T = 100$			$T = 250$			$T = 1000$		
Parameter	Bias	sd	RMSE	Bias	sd	RMSE	Bias	sd	RMSE
τ	−0.026	0.111	0.114	−0.008	0.068	0.069	−0.001	0.035	0.035
Eq.3 Constant	−0.009	0.135	0.135	−0.006	0.081	0.081	0.001	0.040	0.040
Eq.3 Y11_1	−0.001	0.104	0.104	0.001	0.060	0.060	−0.000	0.031	0.031
Eq.3 Y12_1	−0.004	0.103	0.103	−0.000	0.061	0.061	−0.000	0.030	0.030
Eq.3 Y2_1	−0.027	0.125	0.128	−0.011	0.078	0.079	−0.001	0.038	0.038
Eq.1 Constant	−0.002	0.109	0.110	−0.003	0.065	0.065	0.000	0.031	0.031
Eq.1 Y11_1	−0.033	0.090	0.096	−0.012	0.054	0.056	−0.002	0.028	0.028
Eq.1 Y12_1	−0.002	0.096	0.096	0.002	0.057	0.057	0.001	0.027	0.027
Eq.1 Y2_1	−0.000	0.118	0.118	0.001	0.072	0.072	0.000	0.036	0.036
Eq.2 Constant	0.003	0.106	0.106	0.002	0.063	0.063	−0.000	0.031	0.031
Eq.2 Y11_1	0.005	0.091	0.092	0.001	0.056	0.056	0.000	0.027	0.027
Eq.2 Y12_1	−0.026	0.090	0.094	−0.008	0.054	0.055	−0.002	0.028	0.028
Eq.2 Y2_1	−0.001	0.115	0.115	0.000	0.070	0.070	0.002	0.035	0.035
δ_1	0.001	0.114	0.114	0.002	0.071	0.071	0.001	0.035	0.035
δ_2	−0.001	0.116	0.116	−0.002	0.070	0.070	−0.000	0.034	0.034
Ch_11	−0.033	0.069	0.077	−0.012	0.043	0.044	−0.003	0.023	0.023
Ch_21	−0.005	0.103	0.104	−0.001	0.064	0.064	−0.000	0.031	0.031
Ch_22	−0.037	0.068	0.077	−0.014	0.044	0.046	−0.005	0.023	0.024

[a]Computed under DGP1 with $R = 1000$ particles using 1000 MC replications. Parameter names described in Table B.I.

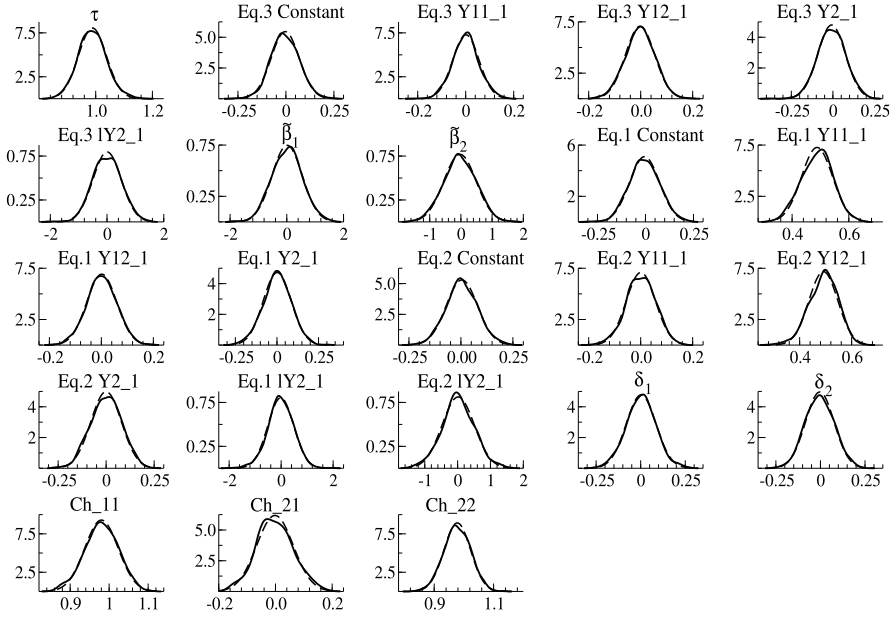

FIGURE F.1.—Sampling densities of ML estimators of reduced-form coefficients of CKSVAR(1) under DGP1 with b chosen such that $\Pr(Y_{2t} = b) = 0.11$ (solid lines) and approximating Normal densities (dashed lines). $T = 250$, 1000 Monte Carlo replications. Parameter names described in Table B.I.

TABLE F.IV

MOMENTS OF SAMPLING DISTRIBUTION OF ML ESTIMATORS OF THE PARAMETERS OF CKSVAR(1).[a]

ML-CKSVAR	True	Mean	Bias	sd	RMSE
τ	1.000	0.988	−0.012	0.050	0.051
Eq.3 Constant	0.000	−0.004	−0.004	0.073	0.073
Eq.3 Y11_1	0.000	0.001	0.001	0.055	0.055
Eq.3 Y12_1	0.000	−0.002	−0.002	0.056	0.056
Eq.3 Y2_1	0.000	−0.008	−0.008	0.083	0.083
Eq.3 lY2_1	0.000	0.003	0.003	0.501	0.501
$\tilde{\beta}_1$	0.000	0.013	0.013	0.533	0.533
$\tilde{\beta}_2$	0.000	−0.030	−0.030	0.518	0.519
Eq.1 Constant	0.000	−0.005	−0.005	0.078	0.078
Eq.1 Y11_1	0.500	0.488	−0.012	0.055	0.056
Eq.1 Y12_1	0.000	0.001	0.001	0.058	0.058
Eq.1 Y2_1	0.000	0.001	0.001	0.084	0.084
Eq.2 Constant	0.000	0.005	0.005	0.075	0.075
Eq.2 Y11_1	0.000	0.001	0.001	0.056	0.056
Eq.2 Y12_1	0.500	0.491	−0.009	0.056	0.056
Eq.2 Y2_1	0.000	−0.000	−0.000	0.080	0.080
Eq.1 lY2_1	0.000	−0.014	−0.014	0.497	0.497
Eq.2 lY2_1	0.000	0.018	0.018	0.491	0.491
δ_1	0.000	0.002	0.002	0.084	0.084
δ_2	0.000	−0.004	−0.004	0.080	0.080
Ch_11	1.000	0.980	−0.020	0.043	0.047
Ch_21	0.000	−0.002	−0.002	0.065	0.065
Ch_22	1.000	0.978	−0.022	0.045	0.050

[a]Computed under DGP1 with b chosen such that $\Pr(Y_{2t} = b) = 0.11$, $T = 250$ using 1000 MC replications. Parameter names described in Table B.I.

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