Econometrica Supplementary Material

SUPPLEMENT TO "OPTIMAL ASSET MANAGEMENT CONTRACTS WITH HIDDEN SAVINGS"

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ONLINE APPENDIX

THIS ONLINE APPENDIX EXTENDS THE RESULTS of Di Tella and Sannikov (2021) to incorporate hidden investment, aggregate risk, and renegotiation. The case with no hidden investment and price of aggregate risk $\pi = 0$ yields the expressions in the paper.

O.1. *Setting With Aggregate Risk and Hidden Investment*

We introduce aggregate risk and hidden investment into the baseline setting in the paper. The observed return is

$$dR_t = (r + \pi\tilde{\sigma} + \alpha - a_t)\,dt + \sigma\,dZ_t + \tilde{\sigma}\,d\tilde{Z}_t, \tag{O.1}$$

where Z and $\tilde{Z}$ are independent Brownian motions that represents idiosyncratic and aggregate risk. There is a complete financial market with equivalent martingale measure Q. The risk-free rate is r, aggregate risk has market price π, and idiosyncratic risk is not priced. Capital has a loading σ on idiosyncratic risk and $\tilde{\sigma}$ on aggregate risk, so the excess return on capital for the agent is α, as in the baseline.

The agent receives cumulative payments I from the principal and manages capital k for him. Payments I can be any semimartingale (it could be decreasing if the agent must pay the principal). This nests the relevant case where the contract gives the agent only what he will consume, that is, $dI_t = c_t\,dt$. As in the baseline setting, the agent can steal from the principal at rate $a_t \geq 0$ and decide when to consume $\tilde{c}_t > 0$. He can invest his hidden savings in the same way the principal would, not only in a risk-free asset, but also in aggregate risk $\tilde{Z}$. In addition, the agent may be able to invest his hidden savings in his private technology. His hidden savings follow the law of motion:

$$dh_t = dI_t + \big(rh_t + z_t h_t(\alpha + \pi\tilde{\sigma}) + \tilde{z}_t h_t \pi - \tilde{c}_t + \phi k_t a_t\big)\,dt + z_t h_t(\sigma\,dZ_t + \tilde{\sigma}\,d\tilde{Z}_t) + \tilde{z}_t h_t\,d\tilde{Z}_t,$$

where z is the portfolio weight on his own private technology, and $\tilde{z}$ the weight on aggregate risk. While the agent can chose any position on aggregate risk, $\tilde{z}_t \in \mathbb{R}$, for his hidden private investment we consider two cases: (1) no hidden private investment, $z_t \in H = \{0\}$, and (2) hidden private investment, $z_t \in H = \mathbb{R}_+$.[22]

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[22]We can also study other cases where the agent may not be able to invest in aggregate risk, or only take a positive position, which requires small modifications to the relevant incentive compatibility constraints. We focus on the economically most relevant case, where the agent can always invest his hidden savings in the market in the same way the principal would.

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The agent's utility is

$$U_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}\, dt\right]$$

and the cost to the principal is

$$J_0 = \mathbb{E}^Q\left[\int_0^\infty e^{-rt}\big(dI_t - (\alpha - a_t)k_t\big)\, dt\right].$$

A contract $\mathcal{C} = (I, k, \tilde{c}, a, z, \tilde{z})$ specifies the contractible payments I and capital k, and recommends the hidden action $(\tilde{c}, a, z, \tilde{z})$, all contingent on the history of observed returns R and the aggregate shock $\tilde{Z}$. After signing the contract the agent can choose a strategy $(\tilde{c}, a, z, \tilde{z})$ to maximize his utility (potentially different from the one recommended by the principal). Given contract $\mathcal{C}$, a strategy is *feasible* if (1) utility $U_0^{\tilde{c},a,z,\tilde{z}}$ is finite, and (2) hidden savings $h_t \geq 0$ always. Since the agent can secretly invest in his private technology, we also impose the regularity condition (3) $\mathbb{E}^Q[\int_0^\infty e^{-rt}(\tilde{c}_t + \alpha z_t h_t)\, dt] < \infty$. Let $\mathbb{S}(\mathcal{C})$ be the set of feasible strategies given contract $\mathcal{C}$.

A contract $\mathcal{C} = (I, k, \tilde{c}, a, z, \tilde{z})$ is *admissible* if (1) $(\tilde{c}, a, z, \tilde{z})$ is feasible given $\mathcal{C}$, and (2)

$$\mathbb{E}^Q\left[\int_0^\infty e^{-rt}\, dI_t\right] < \infty, \qquad \mathbb{E}^Q\left[\int_0^\infty e^{-rt} k_t\, dt\right] < \infty, \quad \mathbb{E}^Q\left[\int_0^\infty e^{-rt} a_t k_t\, dt\right] < \infty. \quad \text{(O.2)}$$

An admissible contract $\mathcal{C} = (I, k, \tilde{c}, a, z, \tilde{z})$ is *incentive compatible* if the agent's optimal feasible strategy given $\mathcal{C}$ is $(\tilde{c}, a, z, \tilde{z})$, as recommended by the principal. Let $\mathbb{IC}$ be the set of incentive compatible contracts. An incentive compatible contract is *optimal* if it minimizes the principal's cost

$$\begin{aligned} v_0 = \min_{\mathcal{C}} J_0(\mathcal{C}) &\equiv \mathbb{E}^Q\left[\int_0^\infty e^{-rt}\big(dI_t - (\alpha - a_t)k_t\big)\, dt\right] \\ \text{s.t.} \quad & U_0^{\tilde{c},a,z,\tilde{z}} \geq u_0 \\ & \mathcal{C} \in \mathbb{IC}. \end{aligned}$$

To incorporate aggregate risk into the setting, we need to slightly modify the parameter restrictions. We assume throughout that

$$\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2 > 0,$$

$$\alpha < \bar{\alpha} \equiv \frac{\phi\sigma\gamma\sqrt{2}}{\sqrt{1+\gamma}}\sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2}.$$

O.2. *No Stealing or Hidden Savings in the Optimal Contract*

LEMMA O.1: *It is without loss of generality to look only at contracts that induce no stealing* $a = 0$, *no hidden savings*, $h = 0$, *and no hidden investment*, $z = \tilde{z} = 0$.

REMARK: This lemma is also valid for the baseline setting without aggregate risk or hidden investment.

PROOF: Imagine the principal is offering contract $\mathcal{C} = (I, k, \tilde{c}, a, z, \tilde{z})$ with associated hidden savings h. Let $k^h = zh$ and $\tilde{k}^h = \tilde{z}h$ be the agent's absolute hidden positions in his private technology and aggregate risk, respectively. We will show that we can offer a new contract $\mathcal{C}' = (I', k', dI', 0, 0, 0)$ under which it is optimal for the agent to choose not to steal, no hidden savings, and no hidden investment, that is, $\tilde{c}' = dI'$, $a = z = \tilde{z} = 0$. The new contract has $I'_t = \int_0^t \tilde{c}_s\, ds$ and $k' = k(R^a) + k^h$ (to simplify notation, we suppress dependence on $\tilde{Z}$).

If the agent now chooses $\tilde{c}' = dI'$, $a = z = \tilde{z} = 0$, he gets hidden savings $h' = 0$ and consumption $\tilde{c}$, so he gets the same utility as under the original contract and this strategy is therefore feasible under the new contract. If instead he chooses a different feasible strategy $(\tilde{c}', a', z', \tilde{z}')$, he gets the utility associated with $\tilde{c}'$. We will show that he could achieve this utility under the original contract by picking consumption $\tilde{c}'$, stealing $dR - dR^a(R^{a'})$, hidden investment in private technology $k^h(R^{a'}) + (k^h)'$, and hidden investment in aggregate risk $\tilde{k}^h(R^{a'}) + (\tilde{k}^h)'$. Since the strategy $(\tilde{c}', a', z', \tilde{z}')$ is feasible under the new contract $\mathcal{C}'$, and $(\tilde{c}, a, z, \tilde{z})$ feasible under the old contract $\mathcal{C}$, then in order to ensure the new strategy is feasible under the original contract we only need to show that hidden savings remain nonnegative always

$$
\begin{aligned}
h'_t = \int_0^t & e^{r(t-s)}\big(dI_t\big(R^a\big(R^{a'}\big)\big) - \tilde{c}'_t\, dt + \phi k_t\big(R^a\big(R^{a'}\big)\big)\big(dR_t - dR^a_t\big(R^{a'}\big)\big) \\
&+ \big(k^h_t\big(R^{a'}\big) + \big(k^h\big)'_t\big)\, dR_t + \big(\tilde{k}^h_t\big(R^{a'}\big) + \big(\tilde{k}^h\big)'_t\big)(\pi\, dt + d\tilde{Z}_t)\big).
\end{aligned}
$$

To show this is always nonnegative, we will show it is greater or equal to the sum of two nonnegative terms. First, the hidden savings under the original contract, following the original feasible strategy, had $R^{a'}$ been the true return

$$
\begin{aligned}
A_t = \int_0^t & e^{r(t-s)}\big(dI_t\big(R^a\big(R^{a'}\big)\big) - \tilde{c}_t\big(R^{a'}\big)\, dt + \phi k_t\big(R^a\big(R^{a'}\big)\big)\big(dR^{a'}_t - dR^a_t\big(R^{a'}\big)\big) \\
&+ k^h_t\big(R^{a'}\big)\, dR^{a'}_t + \tilde{k}^h_t\big(R^{a'}\big)(\pi\, dt + d\tilde{Z}_t)\big) \geq 0.
\end{aligned}
$$

Second, hidden savings under the new contract, following the feasible new strategy:

$$
\begin{aligned}
B_t = \int_0^t & e^{r(t-s)}\big(\tilde{c}_t\big(R^{a'}\big)\, dt - \tilde{c}'_t\, dt + \phi\big(k_t\big(R^a\big(R^{a'}\big)\big) + k^h_t\big(R^{a'}\big)\big)\big(dR_t - dR^{a'}_t\big) \\
&+ \big(k^h\big)'_t\, dR_t + \big(\tilde{k}^h\big)'_t(\pi\, dt + d\tilde{Z}_t)\big) \geq 0.
\end{aligned}
$$

If $\phi = 1$, then $h'_t = A_t + B_t \geq 0$. With $\phi < 1$, we have $h'_t \geq A_t + B_t \geq 0$, because $dR_t - dR^{a'}_t = a'\, dt \geq 0$ and $k^h(R^{a'}) \geq 0$. This means that $\tilde{c}' = c'$, $a = z = \tilde{z} = 0$ is the agent's optimal choice under the new contract $\mathcal{C}'$, since any other choice delivers an utility that he could have obtained—but chose not to—under the original contract $\mathcal{C}$.

We can now compute the principal's cost under the new contract

$$
\begin{aligned}
J'_0 &= \mathbb{E}^Q\bigg[\int_0^{\tau^n} e^{-rt}\big(\tilde{c}_t - \alpha\big(k_t\big(R^a\big) + k^h_t\big)\big)\, dt + e^{-r\tau^n} J'_{\tau^n}\bigg] \\
&= \mathbb{E}^Q\bigg[\int_0^{\tau^n} e^{-rt}\big(dI_t\big(R^a\big) - (\alpha - a_t)k_t\big(R^a\big)\, dt\big) + e^{-r\tau^n} J_{\tau^n}\bigg]
\end{aligned}
$$

$$- \mathbb{E}^Q\left[\int_0^{\tau^n} e^{-rt} a_t k_t(R^a)(1-\phi)\,dt\right]$$
$$- \mathbb{E}^Q\left[\int_0^{\tau^n} e^{-rt}\big(dI_t(R^a) - \tilde{c}_t\,dt + \phi k_t(R^a)a_t\,dt + k_t^h \alpha\,dt\big)\right] + \mathbb{E}^Q\big[e^{-r\tau^n}\big(J'_{\tau^n} - J_{\tau^n}\big)\big].$$

On the rhs, the first term is the cost under the original contract; the second term the destruction produced by stealing under the original contract, which is nonnegative; and the third term is $\mathbb{E}^Q[e^{-r\tau^n} h_{\tau^n}] \geq 0$, where h is the agent's hidden savings under the original contract. To see this, write

$$\begin{aligned} dh_t &= h_t r\,dt + dI_t(R^a) - \tilde{c}_t\,dt + \phi k_t(R^a)a_t\,dt \\ &\quad + k_t^h\big((\alpha + \pi\tilde{\sigma})\,dt + \sigma\,dZ_t + \tilde{\sigma}\,d\tilde{Z}_t\big) + \tilde{k}_t^h(\pi\,dt + d\tilde{Z}_t). \end{aligned}$$

So

$$\begin{aligned} d\big(e^{-rt}h_t\big) &= e^{-rt}\,dh_t - re^{-rt}h_t\,dt \\ &= e^{-rt}\big(dI_t(R^a) - \tilde{c}_t\,dt + \phi k_t(R^a)a_t\,dt \\ &\quad + k_t^h\big((\alpha + \pi\tilde{\sigma})\,dt + \sigma\,dZ_t + \tilde{\sigma}\,d\tilde{Z}_t\big) + \tilde{k}_t^h(\pi\,dt + d\tilde{Z}_t)\big). \end{aligned}$$

Now take expectations under Q, choosing the localizing process appropriately to get

$$\begin{aligned} \mathbb{E}^Q\left[\int_0^{\tau^n} d\big(e^{-rt}h_t\big)\right] &= \mathbb{E}^Q\left[\int_0^{\tau^n} e^{-rt}\big(dI_t(R^a) - \tilde{c}_t\,dt + \phi k_t(R^a)a_t\,dt + k_t^h\alpha\,dt\big)\right] \\ &= \mathbb{E}^Q\big[e^{-r\tau^n}h_{\tau^n} - h_0\big] \geq 0. \end{aligned}$$

Given these inequalities, we can write

$$J'_0 - J_0 \leq \mathbb{E}^Q\big[e^{-r\tau^n}\big(J'_{\tau^n} - J_{\tau^n}\big)\big].$$

Because the original contract was admissible, $\lim_{n\to\infty} \mathbb{E}^Q[e^{-r\tau^n} J_{\tau^n}] = 0$. Since in addition the agent's response was feasible, the new contract is also admissible, and we get $\lim_{n\to\infty} \mathbb{E}^Q[e^{-r\tau^n} J'_{\tau^n}] = 0$ as well. This shows the new contract is admissible, and the cost for the principal is not greater than under the old contract. This completes the proof. *Q.E.D.*

We can then simplify the contract to $\mathcal{C} = (c, k)$, and say an admissible contract is incentive compatible if the agent's optimal strategy is $(c, 0, 0, 0)$, or $(c, 0)$ for short.

O.3. *Incentive Compatibility*

Since the contract can depend on the history of aggregate shocks $\tilde{Z}$, so can his continuation utility $U^{c,0}$ and his consumption c. However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings, his Euler equation needs to be modified appropriately. The discounted marginal utility of a hidden dollar must be a supermartingale under any feasible hidden investment strategy, since otherwise the agent could save a dollar instead of consuming it, invest it in aggregate risk and his private technology, and consume it later when the marginal utility is expected to be higher.

LEMMA O.2: *If $\mathcal{C} = (c, k)$ is an incentive compatible contract, the agent's continuation utility $U^{c,0}$ and consumption c satisfy the laws of motion*

$$dU_t^{c,0} = \left(\rho U_t^{c,0} - \frac{c_t^{1-\gamma}}{1-\gamma}\right) dt + \Delta_t \sigma \, dZ_t + \tilde{\sigma}_t^u \, d\tilde{Z}_t, \tag{O.3}$$

$$\frac{dc_t}{c_t} = \left(\frac{r-\rho}{\gamma} + \frac{1+\gamma}{2}\left(\sigma_t^c\right)^2 + \frac{1+\gamma}{2}\left(\tilde{\sigma}_t^c\right)^2\right) dt + \sigma_t^c \, dZ_t + \tilde{\sigma}_t^c \, d\tilde{Z}_t + dL_t, \tag{O.4}$$

for some Δ, $\tilde{\sigma}^u$, σ^c, $\tilde{\sigma}^c$, and a weakly increasing processes L, such that

$$\Delta_t \geq c_t^{-\gamma} \phi k_t, \tag{O.5}$$

$$z\left(\alpha - \sigma_t^c \sigma \gamma\right) \leq 0 \quad \forall z \in H, \tag{O.6}$$

$$\tilde{\sigma}_t^c = \frac{\pi}{\gamma}. \tag{O.7}$$

PROOF: The proof of (O.3) and (O.5) are similar to Lemmas 1 and 2, where the $d\tilde{Z}$ term appears because the contract can depend on the history of aggregate shocks. For (O.4), the proof is analogous to Lemma 2, but now we need the discounted marginal utility

$$Y_t = e^{\int_0^t r-\rho+z_s(\alpha+\pi\tilde{\sigma})+\pi_s\tilde{z}_s-\frac{1}{2}(z_s\sigma)^2-\frac{1}{2}(z_s\tilde{\sigma}+\tilde{z}_s)^2 \, ds+\int_0^t (z_s\sigma)\, dZ_s+\int_0^t (z_s\tilde{\sigma}+\tilde{z}_s)\, d\tilde{Z}_s} c_t^{-\gamma} \tag{O.8}$$

to be a supermartingale for any investment strategy $\tilde{z}_t \in \mathbb{R}$ and $z_t \in H$. Using the Doob–Meyer decomposition, the Martingale representation theorem, and Ito's lemma, we can write

$$\frac{dc_t}{c_t} = \mu_t^c \, dt + \sigma_t^c \, dZ_t + \tilde{\sigma}_t^c \, d\tilde{Z}_t + dL_t.$$

Since the finite variation part of expression (O.8) must be nonincreasing, we get

$$\begin{aligned} &\left(r - \rho - \gamma\mu_t^c + \frac{\gamma}{2}\left((1+\gamma)\sigma_t^c\right)^2 + \frac{\gamma}{2}\left((1+\gamma)\tilde{\sigma}_t^c\right)^2\right) dt \\ &\quad + \left(z(\alpha + \pi\tilde{\sigma}) + \pi\tilde{z} - \gamma\sigma_t^c z\sigma - \gamma\tilde{\sigma}_t^c(z\tilde{\sigma} + \tilde{z})\right) dt - \gamma \, dL_t \leq 0. \end{aligned} \tag{O.9}$$

Taking $z = \tilde{z} = 0$, which are always allowed, we obtain wlog the expression for μ^c in (O.4), and L weakly increasing. Once we plug this into (O.9), and using that $\tilde{z}_t$ can be both positive or negative, we get (O.7). Condition (O.6) is therefore necessary to ensure (O.9) holds. *Q.E.D.*

The IC constraint (O.6) depends on whether the agent is allowed to have a hidden investment in his own private technology. If hidden investment in the agent's private technology is not allowed, $H = \{0\}$ so condition (O.6) drops out. If instead hidden investment in the agent's private technology is allowed, $H = \mathbb{R}_+$, so condition (O.6) reduces to $\sigma_t^c \geq \frac{\alpha}{\sigma\gamma}$.

O.4. *Change of Variables*

We can still use the the state variables x and $\hat{c}$. Their laws of motion are

$$\frac{dx_t}{x_t} = \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}\left(\sigma_t^x\right)^2 + \frac{\gamma}{2}\left(\tilde{\sigma}_t^x\right)^2\right) dt + \sigma_t^x \, dZ_t + \tilde{\sigma}_t^x \, d\tilde{Z}_t, \tag{O.10}$$

$$\begin{aligned} \frac{d\hat{c}_t}{\hat{c}_t} &= \left(\frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\left(\sigma_t^x\right)^2}{2} + \gamma \sigma_t^x \sigma_t^{\hat{c}} + \frac{1+\gamma}{2}\left(\sigma_t^{\hat{c}}\right)^2\right. \\ &\quad \left. + \frac{\left(\tilde{\sigma}_t^x\right)^2}{2} + \gamma \tilde{\sigma}_t^x \tilde{\sigma}_t^{\hat{c}} + \frac{1+\gamma}{2}\left(\tilde{\sigma}_t^{\hat{c}}\right)^2\right) dt + \sigma_t^{\hat{c}} \, dZ_t + \tilde{\sigma}_t^{\hat{c}} \, d\tilde{Z}_t + dL_t, \\ dL_t &\geq 0 \end{aligned} \tag{O.11}$$

and the incentive compatibility constraints can be written

$$\sigma_t^x \leq \hat{c}_t^{-\gamma} \phi \hat{k}_t \sigma, \tag{O.12}$$

$$z\left(\alpha - \left(\sigma_t^{\hat{c}} + \sigma_t^x\right)\sigma\gamma\right) \leq 0 \quad \forall z \in H, \tag{O.13}$$

$$\tilde{\sigma}_t^{\hat{c}} + \tilde{\sigma}_t^x = \frac{\pi}{\gamma}. \tag{O.14}$$

As before, $\hat{c}$ has an upper bound $\hat{c}_h$, which must be modified to take into account that it is not incentive compatible to give the agent a perfectly safe consumption stream.

LEMMA O.3: *For any incentive compatible contract $\mathcal{C} = (c, k)$, we have for all t*

$$\hat{c}_t \leq \hat{c}_h, \tag{O.15}$$

where $\hat{c}_h$ is given by

$$\begin{aligned} \hat{c}_h &\equiv \max_{\sigma^x \geq 0} \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\sigma^x\right)^2 - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}} \\ &\text{s.t.} \quad z\left(\alpha - \sigma^x \sigma\gamma\right) \leq 0 \quad \forall z \in H. \end{aligned} \tag{O.16}$$

If ever $\hat{c}_t = \hat{c}_h$, then the continuation contract satisfies $\hat{c}_{t+s} = \hat{c}_h$ and $\hat{k}_t = \frac{\sigma^x \hat{c}_h^\gamma}{\phi\sigma}$ for all future times $t+s$, and x_t follows the law of motion (O.10), *where σ^x is the optimizing choice in* (O.16) *and $\tilde{\sigma}^x = \frac{\pi}{\gamma}$. Let $\hat{v}_h$ be the cost of this continuation contract*:

$$\hat{v}_h = \frac{\hat{c}_h - \dfrac{\alpha}{\phi\sigma}\hat{c}_h^\gamma \sigma^x}{r - \dfrac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \dfrac{\gamma}{2}\left(\sigma^x\right)^2 + \dfrac{\gamma}{2}\left(\dfrac{\pi}{\gamma}\right)^2}.$$

If in addition σ^c and $\tilde{\sigma}^c$ are bounded, we have

$$\hat{c}_t \leq \mathbb{E}_t^{\tilde{P}}\left[\int_t^\infty e^{-\int_t^s \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\sigma_u^c)^2 - \frac{1-\gamma}{2}(\tilde{\sigma}_u^c)^2\right) du} \, ds\right]^{-\frac{1}{1-\gamma}}, \tag{O.17}$$

with equality if $L_t = 0$ always, where $\tilde{P}$ is an equivalent measure such that $Z_t - \int_0^t (1-\gamma)\sigma_s^c\, ds$ and $\tilde{Z}_t - \int_0^t (1-\gamma)\tilde{\sigma}_s^c\, ds$ are $\tilde{P}$-martingales.

PROOF: The same reasoning as in Lemma 3 yields $\hat{c}_t \le (\frac{\rho - r(1-\gamma)}{\gamma})^{\frac{1}{1-\gamma}}$. For any $\hat{c}$ between $\hat{c}_h$ and $(\frac{\rho - r(1-\gamma)}{\gamma})^{\frac{1}{1-\gamma}}$, preventing $\hat{c}_t$ from crossing above $\hat{c}$ requires $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$ at that point and $\mu^{\hat{c}} \le 0$. But $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$ implies the drift of $\hat{c}_t$ is

$$\mu_t^{\hat{c}} = \frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{(\sigma_t^x)^2}{2} + \frac{(\tilde{\sigma}_t^x)^2}{2}.$$

If ever $\hat{c}_t > \hat{c}_h$, the drift is strictly positive for any σ_t^x and $\tilde{\sigma}_t^x$ satisfying (O.13) and (O.14), so we must have $\hat{c}_t \le \hat{c}_h$ at all times, and if ever $\hat{c}_t = \hat{c}_h$, it must remain absorbed there forever. Using the IC constraint (O.12) and the law of motion of x, we obtain σ_t^x and $\tilde{\sigma}_t^x$ in the continuation contract, and from (O.12) we get $\hat{k}_t$. The cost of the continuation contract with $\hat{c}_{t+s} = \hat{c}_h$ and $\hat{k}_t = \frac{\sigma^x \hat{c}_h^\gamma}{\phi\sigma}$ for all future times $t+s$ can be obtained from the HJB equation with $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = \mu^{\hat{c}} = 0$, or simply applying the formula (O.32) for the cost of stationary contracts at $\hat{c} = \hat{c}_h$.

For (O.17), the same reasoning as in Lemma 3 gives us

$$\hat{c}_t \le \mathbb{E}_t^{\tilde{P}}\left[\int_t^\infty e^{-\int_t^s (\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\sigma_u^c)^2 - \frac{1-\gamma}{2}(\tilde{\sigma}_u^c)^2)\, du}\, ds\right]^{-\frac{1}{1-\gamma}},$$

with equality if $L_t = 0$ always, where $\tilde{P}$ is an equivalent measure such that $Z_t - \int_0^t (1-\gamma)\sigma_s^c\, ds$ and $\tilde{Z}_t - \int_0^t (1-\gamma)\tilde{\sigma}_s^c\, ds$ are $\tilde{P}$-martingales. *Q.E.D.*

The upper bound $\hat{c}_h$ restricts the principal's ability to promise safety in the future. Even if the agent cannot invest his hidden savings in his private technology, $H = \{0\}$, he can still invest in aggregate risk. In this case, the maximizing choice is $\sigma^c = 0$ and we get $\hat{c}_h = (\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2)^{\frac{1}{1-\gamma}}$. Notice that if $\pi = 0$ this boils down to expression (14) in the baseline setting without aggregate risk or hidden investment. If the agent can also invest his hidden savings in his own private technology, $H = \mathbb{R}_+$, then the maximizing choice is $\sigma^c = \frac{\alpha}{\sigma\gamma}$, and $\hat{c}_h = (\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\alpha}{\sigma\gamma})^2 - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2)^{\frac{1}{1-\gamma}}$ is lower.

We call any admissible contract *locally incentive compatible* if $\hat{c}_t \le \hat{c}_h$, and (O.11)–(O.14) hold. Notice that equation (O.10) follows automatically from the definition of x_t.

We can build a locally incentive compatible contract from processes $x > 0$ and $\hat{c} > 0$ satisfying (O.10), (O.11), (O.13), and (O.14), with $\sigma_t^x \ge 0$ and $\hat{c}_t \le \hat{c}_h$. Define the contract (c, k) by $c_t = \hat{c}_t x_t > 0$ and $k_t = x_t \sigma_t^x \hat{c}_t^\gamma / (\phi\sigma) \ge 0$. Then, under technical conditions, the contract (c, k) is admissible as defined in Section 2 and delivers utility $U_t^{c,0} = \frac{x_t^{1-\gamma}}{1-\gamma}$ under good behavior. It is locally incentive compatible by definition.

LEMMA O.4: *Let $x > 0$ and $\hat{c} > 0$ be stochastic processes satisfying* (O.10), (O.11), (O.13), *and* (O.14), *bounded volatilities $\sigma^x \ge 0$ and $\tilde{\sigma}^x$, and with $\hat{c}$ bounded away from zero and above by $\hat{c}_h$.*

Then the contract $\mathcal{C} = (c, k)$ with $c_t = \hat{c}_t \times x_t > 0$ and $k_t = \sigma_t^x \hat{c}_t^\gamma / (\phi\sigma) \times x_t \ge 0$ delivers utility $U_t^{c,0} = \frac{x_t^{1-\gamma}}{1-\gamma}$ if the agent follows strategy $(c, 0)$. The contract $\mathcal{C}$ is admissible and, therefore, locally incentive compatible if and only if $\mathbb{E}^Q[\int_0^\infty e^{-rt} x_t\, dt] < \infty$.

PROOF: The proof is analogous to Lemma 4 in the main body of the paper. First, we show the contract delivers utility $\frac{x_t^{1-\gamma}}{1-\gamma} < \infty$ if the agent follows strategy $(c, 0)$. Let $Y_t = \frac{x_t^{1-\gamma}}{1-\gamma}$, and using the law of motion of x, (16), we get

$$dY_t = \underbrace{Y_t(1-\gamma)\left(\mu_t^x - \frac{\gamma}{2}(\sigma_t^x)^2 - \frac{\gamma}{2}(\tilde{\sigma}_t^x)^2\right) dt}_{\rho Y_t - \frac{c_t^{1-\gamma}}{1-\gamma}} + Y_t(1-\gamma)\sigma_t^x\, dZ_t + Y_t(1-\gamma)\tilde{\sigma}_t^x\, dZ_t. \tag{O.18}$$

Integrating we obtain

$$Y_0 = \mathbb{E}\left[\int_0^{\tau^n} e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma}\, ds + e^{-\rho\tau^n} Y_{\tau^n}\right]$$

for an increasing sequence of bounded stopping times with $\tau^n \to \infty$ a.s. Take the limit $n \to \infty$, using the monotone convergence theorem on the first term to get

$$Y_0 = \underbrace{\mathbb{E}\left[\int_0^{\infty} e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma}\, ds\right]}_{U_0^{c,0}} + \lim_{n\to\infty} \mathbb{E}\left[e^{-\rho\tau^n} Y_{\tau^n}\right].$$

We will now show that the last term is zero,

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\rho\tau^n} Y_{\tau^n}\right] = 0.$$

Since σ^x and $\tilde{\sigma}^x$ are bounded and $\hat{c}$ bounded away from zero and above by $\hat{c}_h$, μ^x is bounded too, and so is therefore the growth rate, $(1-\gamma)(\mu_t^x - \frac{\gamma}{2}(\sigma_t^x)^2 - \frac{\gamma}{2}(\tilde{\sigma}_t^x)^2)$, and volatilities, $(1-\gamma)\sigma_t^x$ and $(1-\gamma)\tilde{\sigma}_t^x$, of Y_t in (O.18). Furthermore, the growth rate of Y_t in (O.18), is bounded away below ρ,

$$(1-\gamma)\left(\mu_t^x - \frac{\gamma}{2}(\sigma_t^x)^2 - \frac{\gamma}{2}(\tilde{\sigma}_t^x)^2\right) - \rho = -\hat{c}_t^{1-\gamma} \le \max\left\{-\underline{\hat{c}}^{1-\gamma}, -\hat{c}_h^{1-\gamma}\right\} < 0,$$

where $\underline{\hat{c}}$ is a lower bound on $\hat{c}_t$. We then get that $\lim_{n\to\infty} \mathbb{E}[e^{-\rho\tau^n} Y_{\tau^n}] = 0$ and, therefore, $U_0^{c,0} = Y_0 = \frac{x_0^{1-\gamma}}{1-\gamma} < \infty$. The same reasoning yields $U_t^{c,0} = \frac{x_t^{1-\gamma}}{1-\gamma} < \infty$ for all t.

Now we show that the resulting contract $\mathcal{C}$ is admissible if and only if $\mathbb{E}^Q[\int_0^\infty e^{-rt} x_t\, dt] < \infty$. To show sufficiency, notice that since $\hat{c}$ is bounded above by $\hat{c}_h$ and σ^x bounded, we can write

$$\mathbb{E}^Q\left[\int_0^\infty e^{-rt}(c_t + k_t)\, dt\right] \le 2\max\left\{\hat{c}_h, \frac{\bar{\sigma}^x \hat{c}_h^\gamma}{\phi\sigma}\right\}\mathbb{E}^Q\left[\int_0^\infty e^{-rt} x_t\, dt\right] < \infty,$$

where $\bar{\sigma}^x$ is an upper bound on σ_t^x. For necessity, since $\hat{c}_t$ is bounded away from zero, $\hat{c}_t \ge \underline{\hat{c}} > 0$,

$$\infty > \mathbb{E}^Q\left[\int_0^\infty e^{-rt} c_t\, dt\right] \ge \underline{\hat{c}} \times \mathbb{E}^Q\left[\int_0^\infty e^{-rt} x_t\, dt\right].$$

Since the contract is admissible and satisfies (O.11)–(O.14), and $\hat{c}_t \leq \hat{c}_h$, it is locally incentive compatible. *Q.E.D.*

O.5. *Sufficient Conditions for Global Incentive Compatibility*

Incentive compatible contracts are locally incentive compatible. Here, we provide sufficient conditions for a locally incentive compatible contract to be incentive compatible. We can extend Theorem 1 to verify global incentive compatibility.

THEOREM O.1: *Let $\mathcal{C} = (c, k)$ be locally incentive compatible contract with $\hat{c}$ bounded away for zero and bounded volatilities σ^x, $\tilde{\sigma}^x$, $\sigma^{\hat{c}}$ and $\tilde{\sigma}^{\hat{c}}$. Suppose that the contract satisfies the following property*:

$$\sigma_t^{\hat{c}} \leq 0.$$

Then for any feasible strategy $(\tilde{c}, a, z, \tilde{z})$, with associated hidden savings h, we have the following upper bound on the agent's utility, after any history:

$$U_t^{\tilde{c},a,z,\tilde{z}} \leq \left(1 + \frac{h_t}{x_t}\hat{c}_t^{-\gamma}\right)^{1-\gamma} U_t^{c,0}.$$

In particular, since $h_0 = 0$, for any feasible strategy $U_0^{\tilde{c},a,z,\tilde{z}} \leq U_0^{c,0}$, and the contract $\mathcal{C}$ is therefore incentive compatible.

PROOF: Focus on the simple case with $dL_t = 0$; the proof can be easily generalized for $dL_t \geq 0$. Following the same steps as in the proof of Theorem 1, we obtain

$$\begin{aligned}
&e^{-\rho t}\left(U_t^{\tilde{c},a} - \frac{\bar{x}_t^{1-\gamma}}{1-\gamma}\right) \\
&\quad = \mathbb{E}_t^a\left[\int_t^{\tau^n} e^{-\rho u}\frac{\breve{c}_u^{1-\gamma}}{1-\gamma}\,du + \int_t^{\tau^n} d\left(e^{-\rho u}\frac{\bar{x}_u^{1-\gamma}}{1-\gamma}\right) + e^{-\rho\tau^n}\left(U_{\tau^n}^{\tilde{c},a} - \frac{\bar{x}_{\tau^n}^{1-\gamma}}{1-\gamma}\right)\right],
\end{aligned}$$

where $\bar{x}_t = x_t + h_t\hat{c}_t^{-\gamma}$, and $\breve{c}_t = \tilde{c}_t/\bar{x}_t$. The first two terms can be written

$$\begin{aligned}
&\mathbb{E}_t^a\left[\int_t^{\tau^n} e^{-\rho u}\frac{\breve{c}_u^{1-\gamma}}{1-\gamma}\,du + \int_t^{\tau^n} d\left(e^{-\rho u}\frac{\bar{x}_u^{1-\gamma}}{1-\gamma}\right)\right] \\
&\quad = \mathbb{E}_t^a\left[\int_t^{\tau^n} e^{-\rho u}\bar{x}_u^{-\gamma}\left(\frac{\breve{c}_u^{1-\gamma} - \rho}{1-\gamma}\bar{x}_u + \bar{x}_u\mu_u^{\bar{x}} - \bar{x}_u\frac{\gamma}{2}\left(\sigma_u^{\bar{x}}\right)^2 - \bar{x}_u\frac{\gamma}{2}\left(\tilde{\sigma}_u^{\bar{x}}\right)^2\right)du\right],
\end{aligned} \tag{O.19}$$

where

$$\begin{aligned}
\bar{x}_t\mu_t^{\bar{x}} &= x_t\left(\frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}\left(\sigma_t^x\right)^2 + \frac{\gamma}{2}\left(\tilde{\sigma}_t^x\right)^2 - \frac{\sigma_t^x}{\sigma}a_t\right) \\
&\quad + \hat{c}_t^{-\gamma}\left(rh_t + z_th_t(\alpha + \pi\tilde{\sigma}) + \tilde{z}_th_t\pi + c_t - \tilde{c}_t + \phi k_ta_t\right) \\
&\quad + h_t\hat{c}_t^{-\gamma}\left(\rho - r - \gamma\frac{\hat{c}^{1-\gamma} - \rho}{1-\gamma} - \frac{\gamma}{2}\left(\sigma_t^x\right)^2 - \gamma^2\sigma_t^x\sigma_t^{\hat{c}} - \frac{\gamma}{2}\left(\tilde{\sigma}_t^x\right)^2 - \gamma^2\tilde{\sigma}_t^x\tilde{\sigma}_t^{\hat{c}} + \gamma\sigma_t^{\hat{c}}a_t\right) \\
&\quad - h_t\hat{c}_t^{-\gamma}\gamma\sigma_t^{\hat{c}}z_t\sigma - h_t\hat{c}_t^{-\gamma}\gamma\tilde{\sigma}_t^{\hat{c}}(z_t\tilde{\sigma} + \tilde{z}_t),
\end{aligned}$$

$$\bar{x}_t\sigma_t^{\bar{x}} = \sigma_t^x x_t - h_t\hat{c}_t^{-\gamma}\gamma\sigma_t^{\hat{c}} + h_t\hat{c}_t^{-\gamma}z_t\sigma,$$
$$\bar{x}_t\tilde{\sigma}_t^{\bar{x}} = \tilde{\sigma}_t^x - h_t\hat{c}_t^{-\gamma}\gamma\tilde{\sigma}_t^{\hat{c}} + h_t\hat{c}_t^{-\gamma}(z_t\tilde{\sigma} + \tilde{z}_t).$$

Following the proof of Theorem 1, we can write the integrand as the sum of four parts

$$\frac{\check{c}_u^{1-\gamma} - \rho}{1-\gamma}\bar{x}_u + \bar{x}_u\mu_u^{\bar{x}} - \bar{x}_u\frac{\gamma}{2}\left(\sigma_u^{\bar{x}}\right)^2 - \bar{x}_u\frac{\gamma}{2}\left(\tilde{\sigma}_u^{\bar{x}}\right)^2 = A_u a_u + B_u + C_u + \tilde{C}_u.$$

A_t and B_t are unchanged and we know they are nonpositive:

$$A_t = \hat{c}_t^{-\gamma}\phi k_t - x_t\frac{\sigma_t^x}{\sigma} + h_t\hat{c}_t^{-\gamma}\gamma\sigma_t^{\hat{c}} \le 0,$$
$$B_t = x_t\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \hat{c}_t^{-\gamma}(rh_t + \hat{c}_t x_t - \check{c}_t\bar{x}_t) + h_t\hat{c}_t^{-\gamma}\left(\rho - r - \gamma\frac{\hat{c}^{1-\gamma} - \rho}{1-\gamma}\right) + \bar{x}_t\frac{\check{c}_u^{1-\gamma} - \rho}{1-\gamma} \le 0.$$

C_t needs to be modified to account for hidden investment, and the new term $\tilde{C}_t$ collects the terms dealing with aggregate risk,

$$C_t = x_t\frac{\gamma}{2}\left(\sigma_t^x\right)^2 + h_t\hat{c}_t^{-\gamma}\left(z_t\alpha - \gamma\sigma_t^{\hat{c}}z_t\sigma - \frac{\gamma}{2}\left(\sigma_t^x\right)^2 - \gamma^2\sigma_t^x\sigma_t^{\hat{c}}\right) - \bar{x}_t\frac{\gamma}{2}\left(\sigma_t^{\bar{x}}\right)^2,$$
$$C_t = \frac{\gamma}{2}\left(x_t\left(\sigma_t^x\right)^2 + 2h_t\hat{c}_t^{-\gamma}z_t\left(\frac{\alpha}{\gamma} - \sigma_t^{\hat{c}}\sigma\right) - h_t\hat{c}_t^{-\gamma}\left(\sigma_t^x\right)^2 - h_t\hat{c}_t^{-\gamma}2\gamma\sigma_t^x\sigma_t^{\hat{c}} - \frac{1}{\bar{x}_t}\left(\bar{x}_t\sigma_t^{\bar{x}}\right)^2\right).$$

Notice that we included the $z_t h_t\alpha$ term here. We will include $\pi(z_t h_t\tilde{\sigma} + \tilde{z}_t h_t)$ in $\tilde{C}_t$.

Expand $(\bar{x}_t\sigma_t^{\bar{x}})^2$:

$$C_t = \frac{\gamma}{2}\left(x_t\left(\sigma_t^x\right)^2 + 2h_t\hat{c}_t^{-\gamma}z_t\left(\frac{\alpha}{\gamma} - \sigma_t^{\hat{c}}\sigma\right) - h_t\hat{c}_t^{-\gamma}\left(\sigma_t^x\right)^2 - h_t\hat{c}_t^{-\gamma}2\gamma\sigma_t^x\sigma_t^{\hat{c}}\right.$$
$$\left. - \frac{1}{\bar{x}_t}\left(x_t^2\left(\sigma_t^x\right)^2 + \left(h_t\hat{c}_t^{-\gamma}\left(z_t\sigma - \gamma\sigma_t^{\hat{c}}\right)\right)^2 + 2x_t\sigma_t^x h_t\hat{c}_t^{-\gamma}\left(z_t\sigma - \gamma\sigma_t^{\hat{c}}\right)\right)\right).$$

Take the $1/\bar{x}_t$ out of the parenthesis:

$$C_t = \frac{\gamma}{2}\frac{1}{\bar{x}_t}\left(x_t^2\left(\sigma_t^x\right)^2 + h_t\hat{c}_t^{-\gamma}x_t\left(\sigma_t^x\right)^2 + 2x_t h_t\hat{c}_t^{-\gamma}z_t\left(\frac{\alpha}{\gamma} - \sigma_t^{\hat{c}}\sigma\right) + 2\left(h_t\hat{c}_t^{-\gamma}\right)^2 z_t\left(\frac{\alpha}{\gamma} - \sigma_t^{\hat{c}}\sigma\right)\right.$$
$$- x_t h_t\hat{c}_t^{-\gamma}\left(\sigma_t^x\right)^2 - \left(h_t\hat{c}_t^{-\gamma}\right)^2\left(\sigma_t^x\right)^2 - x_t h_t\hat{c}_t^{-\gamma}2\gamma\sigma_t^x\sigma_t^{\hat{c}} - \left(h_t\hat{c}_t^{-\gamma}\right)^2 2\gamma\sigma_t^x\sigma_t^{\hat{c}}$$
$$\left. - \left(x_t^2\left(\sigma_t^x\right)^2 + \left(h_t\hat{c}_t^{-\gamma}\left(z_t\sigma - \gamma\sigma_t^{\hat{c}}\right)\right)^2 + 2x_t\sigma_t^x h_t\hat{c}_t^{-\gamma}\left(z_t\sigma - \gamma\sigma_t^{\hat{c}}\right)\right)\right).$$

Cancel some terms:

$$C_t = \frac{\gamma}{2}\frac{1}{\bar{x}_t}\left(2x_t h_t\hat{c}_t^{-\gamma}z_t\left(\frac{\alpha}{\gamma} - \sigma_t^{\hat{c}}\sigma\right) + 2\left(h_t\hat{c}_t^{-\gamma}\right)^2 z_t\left(\frac{\alpha}{\gamma} - \sigma_t^{\hat{c}}\sigma\right) - \left(h_t\hat{c}_t^{-\gamma}\right)^2\left(\sigma_t^x\right)^2\right.$$
$$\left. - \left(h_t\hat{c}_t^{-\gamma}\right)^2 2\gamma\sigma_t^x\sigma_t^{\hat{c}} - \left(h_t\hat{c}_t^{-\gamma}\left(z_t\sigma - \gamma\sigma_t^{\hat{c}}\right)\right)^2 - 2x_t\sigma_t^x h_t\hat{c}_t^{-\gamma}z_t\sigma\right).$$

And group the remaining ones to form a square:

$$C_t = -\frac{\gamma}{2}\frac{\left(h_t\hat{c}_t^{-\gamma}\right)^2}{\bar{x}_t}\left(\left(\sigma_t^x\right)^2 + \left(\gamma\sigma_t^{\hat{c}} - z_t\sigma\right)^2 + 2\sigma_t^x\left(\gamma\sigma_t^{\hat{c}} - z_t\sigma\right)\right)$$
$$+ \frac{\gamma}{2}\frac{1}{\bar{x}_t}\left(2x_t h_t\hat{c}_t^{-\gamma}z_t\left(\frac{\alpha}{\gamma} - \left(\sigma_t^x + \sigma_t^{\hat{c}}\right)\sigma\right) + 2\left(h_t\hat{c}_t^{-\gamma}\right)^2 z_t\left(\frac{\alpha}{\gamma} - \left(\sigma_t^x + \sigma_t^{\hat{c}}\right)\sigma\right)\right),$$
$$C_t = -\frac{\gamma}{2}\frac{\left(h_t\hat{c}_t^{-\gamma}\right)^2}{\bar{x}_t}\left(\sigma_t^x + \left(\gamma\sigma_t^{\hat{c}} - z_t\sigma\right)\right)^2 + \frac{1}{\bar{x}_t}\left(x_t + h_t\hat{c}_t^{-\gamma}\right)h_t\hat{c}_t^{-\gamma}z_t\left(\alpha - \gamma\left(\sigma_t^x + \sigma_t^{\hat{c}}\right)\sigma\right).$$

Rearrange

$$C_t = -\frac{\gamma}{2}\frac{\left(h_t\hat{c}_t^{-\gamma}\right)^2}{\bar{x}_t}\left(\sigma_t^x + \left(\gamma\sigma_t^{\hat{c}} - z_t\sigma\right)\right)^2 + h_t\hat{c}_t^{-\gamma}z_t\sigma\left(\frac{\alpha}{\sigma} - \gamma\left(\sigma_t^x + \sigma_t^{\hat{c}}\right)\right) \leq 0.$$

where the last inequality uses the IC constraint for hidden investment (O.13).

The term $\tilde{C}_t$ collects the terms dealing with aggregate risk:

$$\tilde{C}_t = x_t\frac{\gamma}{2}\left(\tilde{\sigma}_t^x\right)^2 + h_t\hat{c}_t^{-\gamma}\left((z_t\tilde{\sigma} + \tilde{z}_t)\pi - \gamma\tilde{\sigma}_t^{\hat{c}}(z_t\tilde{\sigma} + \tilde{z}_t) - \frac{\gamma}{2}\left(\tilde{\sigma}_t^x\right)^2 - \gamma^2\tilde{\sigma}_t^x\tilde{\sigma}_t^{\hat{c}}\right) - \bar{x}_t\frac{\gamma}{2}\left(\tilde{\sigma}_t^{\bar{x}}\right)^2.$$

This term is the same as C_t except that α/σ is replaced with π and $z_t\sigma$ is replaced with $z_t\tilde{\sigma} + \tilde{z}_t$, and all the volatilities are with respect to the aggregate shock. The same steps as above therefore lead to

$$\tilde{C}_t = -\frac{\gamma}{2}\frac{\left(h_t\hat{c}_t^{-\gamma}\right)^2}{\bar{x}_t}\left(\tilde{\sigma}_t^x + \left(\gamma\tilde{\sigma}_t^{\hat{c}} - (z_t\tilde{\sigma} + \tilde{z}_t)\right)\right)^2 + h_t\hat{c}_t^{-\gamma}(z_t\tilde{\sigma} + \tilde{z}_t)\left(\pi - \gamma\left(\tilde{\sigma}_t^x + \tilde{\sigma}_t^{\hat{c}}\right)\right) \leq 0,$$

where the last inequality follows from the IC constraint for hidden investment in aggregate risk (O.14).

The rest of the proof dealing with the terminal term follows the same steps as in Theorem 1 using Lemma O.14. *Q.E.D.*

O.6. *The Solution to the Relaxed Problem Gives the Optimal Contract*

The *relaxed problem* minimizes cost within the class of locally incentive compatible contracts, and we call a solution to the relaxed problem a *relaxed optimal contract*. As in the baseline setting, the relaxed optimal contract is in fact globally incentive compatible and, therefore, an optimal contract. The solution to the relaxed problem can be characterized with the same HJB equation as in the case without hidden investment, appropriately extended to incorporate aggregate risk and the new incentive compatibility constraints:

$$0 = \min_{\sigma^x,\sigma^{\hat{c}},\tilde{\sigma}^x,\tilde{\sigma}^{\hat{c}}} \hat{c} - r\hat{v} - \sigma^x\hat{c}^\gamma\frac{\alpha}{\phi\sigma} + \hat{v}\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}\left(\sigma^x\right)^2 + \frac{\gamma}{2}\left(\tilde{\sigma}^x\right)^2 - \pi\tilde{\sigma}^x\right)$$
$$+ \hat{v}'\hat{c}\left(\frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\left(\sigma^x\right)^2}{2} + (1+\gamma)\sigma^x\sigma^{\hat{c}} + \frac{1+\gamma}{2}\left(\sigma^{\hat{c}}\right)^2 + \frac{\left(\tilde{\sigma}^x\right)^2}{2}\right.$$
$$\left.+ (1+\gamma)\tilde{\sigma}^x\tilde{\sigma}^{\hat{c}} + \frac{1+\gamma}{2}\left(\tilde{\sigma}^{\hat{c}}\right)^2 - \tilde{\sigma}^{\hat{c}}\pi\right) + \frac{\hat{v}''}{2}\hat{c}^2\left(\left(\sigma^{\hat{c}}\right)^2 + \left(\tilde{\sigma}^{\hat{c}}\right)^2\right), \tag{O.20}$$

subject to $\sigma^x \geq 0$ and (O.13) and (O.14).

Using (O.14) to eliminate $\tilde{\sigma}^{\hat{c}}$, and taking FOC for $\tilde{\sigma}^{x}$, we obtain

$$\tilde{\sigma}^{x} = \frac{\pi}{\gamma}, \qquad \tilde{\sigma}^{\hat{c}} = 0.$$

This is the first best exposure to aggregate risk. The principal and the agent do not have any conflict about aggregate risk, and the principal cannot use it to relax the moral hazard problem, so they implement the first best aggregate risk sharing.[23]

The FOC for σ^{x} and $\sigma^{\hat{c}}$ depend on whether the agent can invest his hidden savings in his private technology. Without hidden investment, the FOCs are the same as in the baseline. With hidden investment, the IC constraint (O.13) could be binding in some region of the state space. The shape of the contract, however, is the same as in the baseline without hidden investment.

It is useful to define

$$A(\hat{c}, \hat{v}) \equiv \min_{\sigma^{x} \geq 0} \hat{c} - \sigma^{x} \hat{c}^{\gamma} \frac{\alpha}{\phi \sigma} - r\hat{v} + \hat{v}\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^{x})^{2} - \frac{\gamma}{2}\left(\frac{\pi}{\gamma}\right)^{2}\right). \tag{O.21}$$

The HJB equation when $\hat{v}' = \hat{v}'' = 0$ and (O.13) is not binding is $A(\hat{c}, \hat{v}) = 0$.

THEOREM O.2: *The relaxed problem has the following properties*:

1. *The cost function $\hat{v}(\hat{c})$ has a flat portion on $(0, \hat{c}_l)$ and a strictly increasing C^2 portion on $(\hat{c}_l, \hat{c}_h)$, for some $\hat{c}_l \in (0, \hat{c}_h)$. The HJB equation* (O.20) *holds with equality in $\hat{c} \in (\hat{c}_l, \hat{c}_h)$. For $\hat{c} < \hat{c}_l$, we have $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_l) \equiv \hat{v}_l$ and the HJB holds as an inequality, $A(\hat{c}, \hat{v}_l) > 0$.*
2. *At $\hat{c}_l$, we have $\hat{v}'(\hat{c}_l) = 0$, $\hat{v}''_{+}(\hat{c}_l) > 0$, and $A(\hat{c}_l, \hat{v}_l) = 0$. The cost function satisfies $\hat{v}(\hat{c}) < \hat{c}^{\gamma}$ for all $\hat{c} \in [\hat{c}_l, \hat{c}_h)$, with $\hat{v}(\hat{c}_h) = \hat{v}_h$.*
3. *The state variables x_t and $\hat{c}_t$ follow the laws of motion* (O.10) *and* (O.11) *with bounded $\sigma^{x}_{t} > 0$, $\sigma^{\hat{c}}_{t} < 0$, $\tilde{\sigma}^{x}_{t} = \frac{\pi}{\gamma}$, and $\tilde{\sigma}^{\hat{c}}_{t} = 0$ for all $t > 0$, and $dL_t = 0$ always, so the Euler equation holds as an equality. The state $\hat{c}_t$ starts at $\hat{c}_0 = \hat{c}_l$, with $\mu^{\hat{c}}_{0} > 0$ and $\sigma^{\hat{c}}_{0} = \tilde{\sigma}^{\hat{c}}_{0} = 0$, and immediately moves into the interior of the domain never reaching either boundary, that is, $\hat{c}_t \in (\hat{c}_l, \hat{c}_h)$ for all $t > 0$.*
4. *Without hidden investment, the optimal contract in the relaxed problem does not have a stationary distribution*:

$$\frac{1}{t}\int_{0}^{t} 1_{\{\hat{c}_t > \hat{c}_h - \epsilon\}}(\hat{c}_s)\, ds \to 1 \quad a.s.\ \forall \epsilon > 0,$$

but $P\{\hat{c}_t \to \hat{c}_h\} = 0$. With hidden investment, the optimal contract in the relaxed problem has a stationary distribution with density proportional to

$$m(\hat{c}) = \frac{1}{\sigma^{\hat{c}}(\hat{c})\hat{c}} \exp\left(\int^{\hat{c}} \frac{2\mu^{\hat{c}}(z)z}{(\sigma^{\hat{c}}(z)z)^{2}}\, dz\right),$$

which spikes near $\hat{c}_h$, that is, $m(\hat{c}) \to \infty$ as $\hat{c} \to \hat{c}_h$.

[23] If the agent did not have access to hidden investment in aggregate risk, and the agent's private technology is exposed to aggregate risk $\tilde{\sigma} \neq 0$, then the principal could potentially use the agent's exposure to aggregate risk to relax the moral hazard problem.

5. *Since the relaxed optimal contract satisfies the sufficient condition in Theorem* O.1, *it is incentive compatible and*, *therefore*, *an optimal contract*.

PROOF: The proof is similar to Theorem 2, except we use the more general definition of $A(\hat{c}, \hat{v})$ in (O.21), which once we optimize over σ^x can be written

$$A(\hat{c}, \hat{v}) = \hat{c} - r\hat{v} - \frac{1}{2}\frac{\left(\frac{\hat{c}^\gamma \alpha}{\phi\sigma}\right)}{\hat{v}\gamma} + \hat{v}\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma}\right).$$

Notice we already know from the FOCs that $\tilde{\sigma}^x = \pi/\gamma$ and $\tilde{\sigma}^{\hat{c}} = 0$.

Part (1) and *part (2)* go through without modifications.

In *Part (3)*, the proof that $A(\hat{c}_l, \hat{v}(\hat{c}_l)) = 0$ requires that we consider the possibility that the hidden investment constraint is binding as we approach $\hat{c}_l$. Since the right-hand side of the HJB can only be greater if the hidden investment constraint is binding, we get that $A(\hat{c}_l, \hat{v}(\hat{c}_l)) \le 0$. But we know that $A(\hat{c}, \hat{v}(\hat{c}_l)) \ge 0$ for $\hat{c} < \hat{c}_l$ from Lemma O.11, and since A is continuous in $\hat{c}$, we get that $A(\hat{c}_l, \hat{v}(\hat{c}_l)) = 0$, as desired.

Part 4 goes through with natural modifications. The first-order ODE has the natural modification

$$\underbrace{\hat{c} - rf - \sigma^x\hat{c}^\gamma\frac{\alpha}{\phi\sigma} + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}\left(\sigma^x\right)^2 - \frac{1}{2}\frac{\pi^2}{\gamma}\right)}_{A(\hat{c},f)} + f'\hat{c}\left(\frac{\hat{c}^{1-\gamma} - (\hat{c}_u)^{1-\gamma}}{1-\gamma} + \frac{\left(\sigma^x\right)^2}{2}\right) = 0,$$

where we are fixing $\sigma^{\hat{c}} = 0$ and $\sigma^x = \frac{\alpha}{\sigma\gamma}\frac{\hat{c}_l^\gamma}{\hat{v}_l\phi}$. This is consistent with the hidden investment constraint because $\hat{c}_l^\gamma \ge \hat{v}_l$ from Lemma O.15. The rest of the proof goes though and we get $\hat{v}''_+(\hat{c}_l) > 0$.

Part (5) and (6) go though with appropriate modifications, noticing that $\sigma^{\hat{c}}(\hat{c}_l) = 0$ and $\sigma^x(\hat{c}_l) = \frac{\alpha}{\sigma\gamma}\frac{\hat{c}_l^\gamma}{\hat{v}_l}$ satisfy the hidden investment constraint because $\hat{v}_l \le \hat{c}_l^\gamma$, from Lemma O.15. The proof that $\hat{v}(\hat{c}) < \hat{c}^\gamma$ for $\hat{c} \in (\hat{c}_l, \hat{c}_h)$ is unchanged, except replacing $\hat{c}_h$ with $\hat{c}_u$ where appropriate. We only need to check that $\sigma^{\hat{c}}(\hat{c}) < 0$ and $\sigma^x(\hat{c}) > 0$ for all $\hat{c} \in (\hat{c}_l, \hat{c}_h)$ in the case where the hidden investment constraint is binding. In that case $\sigma^{\hat{c}} = \frac{\alpha}{\sigma\gamma} - \sigma^x$, and the FOC for σ^x yields

$$\sigma^x = \frac{\hat{c}^\gamma\frac{\alpha}{\phi\sigma} + \hat{v}''\hat{c}^2\frac{\alpha}{\sigma\gamma}}{\gamma\left(\hat{v} - \hat{v}'\hat{c}\right) + \hat{v}''\hat{c}^2} > \frac{\alpha}{\sigma\gamma},$$

which implies $\sigma^{\hat{c}} < 0$. To see this inequality, use $\hat{v} < \hat{c}^\gamma$ to write

$$\phi\hat{c}^{-\gamma}\hat{v} < 1,$$

and use $\hat{v}' \ge 0$ to get

$$\phi\hat{c}^{-\gamma}\left(\hat{v} - \hat{v}'\hat{c}\right) < 1,$$

$$\frac{\alpha}{\sigma}\left(\hat{v} - \hat{v}'\hat{c}\right) < \hat{c}^\gamma\frac{\alpha}{\phi\sigma},$$

$$\frac{\alpha}{\gamma\sigma}\left(\gamma\left(\hat{v} - \hat{v}'\hat{c}\right) + \hat{v}''\hat{c}^2\right) < \hat{c}^\gamma\frac{\alpha}{\phi\sigma} + \hat{v}''\hat{c}^2\frac{\alpha}{\gamma\sigma}.$$

Finally, divide throughout by $\gamma(\hat{v}-\hat{v}'\hat{c})+\hat{v}''\hat{c}^2$ which must be strictly positive (second-order condition for optimality)

$$\frac{\alpha}{\gamma\sigma}<\frac{\hat{c}^{\gamma}\dfrac{\alpha}{\phi\sigma}+\hat{v}''\hat{c}^2\dfrac{\alpha}{\gamma\sigma}}{\gamma\big(\hat{v}-\hat{v}'\hat{c}\big)+\hat{v}''\hat{c}^2}=\sigma^x.$$

Part (7) is unchanged for the behavior near $\hat{c}_l$. For $\hat{c}_h$, we need to consider two cases. Without hidden investment, Lemma O.17 shows that

$$\mu^{\hat{c}}\hat{c}\approx\big(4\gamma-6(1+\gamma)^2\big)\hat{c}_h^{-\gamma}\epsilon,$$
$$\sigma^{\hat{c}}\hat{c}\approx-\sqrt{2}2(1+\gamma)\hat{c}_h^{-\gamma/2}\epsilon^{3/2},$$

and the same analysis as in Theorem 2 shows that $\hat{c}_h$ is inaccessible. With hidden investment, the IC constraint will be binding near the upper boundary. Lemma O.17 shows that

$$\mu^{\hat{c}}\hat{c}\approx(\eta-2)\frac{1}{2}\left(\frac{\alpha}{\sigma\gamma}\right)^2\left(\frac{\gamma}{1-\eta}\right)^2(\hat{c}_h-\hat{c})<0,$$
$$\sigma^{\hat{c}}\hat{c}\approx-\left(\frac{\alpha}{\sigma\gamma}\right)\frac{\gamma}{1-\eta}(\hat{c}_h-\hat{c}),$$

for some $\eta\in(0,1)$. We can compute the scale function

$$S(\hat{c})=\int^{\hat{c}}\exp\left(-\int^{y}\frac{2\bar{\mu}}{\bar{\sigma}^2}\frac{1}{\hat{c}_h-z}\,dz\right)dy=-\frac{1}{2\bar{\mu}/\bar{\sigma}^2+1}(\hat{c}_h-\hat{c})^{\frac{2\bar{\mu}}{\bar{\sigma}^2}+1},$$

where $\bar{\mu}=(\eta-2)\frac{1}{2}(\frac{\alpha}{\sigma\gamma})^2(\frac{\gamma}{1-\eta})^2<0$ and $\bar{\sigma}^2=(\frac{\alpha}{\sigma\gamma})^2(\frac{\gamma}{1-\eta})^2$, so that $2\bar{\mu}/\bar{\sigma}^2=\eta-2<-1$. So $S(\hat{c}_h)=\infty$, which means that $\hat{c}_h$ is inaccessible and nonattracting ($P\{\hat{c}_t\to\hat{c}_h\}=0$).

For *Part (8)*, without hidden investment the proof is unchanged. For the case with hidden investment, for the behavior near $\hat{c}_h$ we must compute the speed measure

$$m(\hat{c})=\frac{1}{\bar{\sigma}^2(\hat{c}_h-\hat{c})^2}\exp\left(\int^{\hat{c}}\frac{2\bar{\mu}}{\bar{\sigma}^2}\frac{1}{\hat{c}_h-z}\,dz\right).$$

Using the approximation,

$$\mu^{\hat{c}}\hat{c}\approx(\eta-2)\frac{1}{2}\left(\frac{\alpha}{\sigma\gamma}\right)^2\left(\frac{\gamma}{1-\eta}\right)^2(\hat{c}_h-\hat{c})<0,$$
$$\sigma^{\hat{c}}\hat{c}\approx-\left(\frac{\alpha}{\sigma\gamma}\right)\frac{\gamma}{1-\eta}(\hat{c}_h-\hat{c}),$$

we get that near $\hat{c}_h$

$$m(\hat{c})\approx\frac{1}{\bar{\sigma}^2}(\hat{c}_h-\hat{c})^{-\frac{2\bar{\mu}}{\bar{\sigma}^2}-2},$$

where $-\frac{2\bar{\mu}}{\bar{\sigma}^2} - 2 = 2 - \eta - 2 = -\eta < 0$. This means that $m(\hat{c}) \to \infty$ as $\hat{c} \to \hat{c}_h$. But the integral of $m(\hat{c})$ is

$$M(\hat{c}) = \int^{\hat{c}} \frac{1}{\bar{\sigma}^2}(\hat{c}_h - z)^{-\eta}\, dz = \frac{1}{1-\eta}\frac{1}{\bar{\sigma}^2}(\hat{c}_h - \hat{c})^{1-\eta},$$

which is finite as $\hat{c} \to \hat{c}_h$. Since $\hat{c}_l$ is an entrance boundary, we have a stationary distribution,

$$\psi(\hat{c}) = \frac{m(\hat{c})}{\displaystyle\int_{\hat{c}_l}^{\hat{c}_h} m(z)\, dz},$$

with a spike near $\hat{c}_h$.

Part (9) uses the more general Theorem O.1. *Q.E.D.*

For a given solution to the HJB equation, we can identify controls σ^x and $\sigma^{\hat{c}}$ as functions of $\hat{c}$, and use those to build a candidate optimal contract $\mathcal{C}^*$. Specifically, let x^* and $\hat{c}^*$ be the solutions to (O.10) and (O.11) with $\sigma_t^{x^*} = \sigma^x(\hat{c}_t^*)$, $\sigma_t^{\hat{c}^*} = \sigma^{\hat{c}}(\hat{c}_t^*)$ and $dL_t = 0$, starting from initial values $x_0^* = ((1-\gamma)u_0)^{\frac{1}{1-\gamma}}$ and $\hat{c}_0^* = \hat{c}_l$. We then construct the candidate contract $\mathcal{C}^* = (c^*, k^*)$ with $c^* = \hat{c}^* x^*$ and $k^* = \sigma^{x^*}\frac{(\hat{c}^*)^\gamma}{\phi\sigma}x^*$.

THEOREM O.3: *Let $\hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \to [\hat{v}_l, \hat{v}_h]$ be a strictly increasing C^2 solution to the HJB equation (O.20) for some $\hat{c}_l \in (0, \hat{c}_h)$, such that $\hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h]$, $\hat{v}'(\hat{c}_l) = 0$, $\hat{v}''(\hat{c}_l) > 0$ and $\hat{v}(\hat{c}_h) = \hat{v}_h$. Assume that for $\hat{c} < \hat{c}_l$ the HJB equation holds as an inequality, $A(\hat{c}, \hat{v}_l) > 0$, and that $\hat{v}(\hat{c}) \le \hat{c}^\gamma$ for $\hat{c} \in [\hat{c}_l, \hat{c}_h]$. Then*:

1. *For any locally incentive compatible contract $\mathcal{C} = (c, k)$ that delivers at least utility u_0 to the agent, we have $\hat{v}(\hat{c}_l)((1-\gamma)u_0)^{\frac{1}{1-\gamma}} \le J_0(\mathcal{C})$.*
2. *Let $\mathcal{C}^*$ be a candidate optimal contract generated by the policy functions of the HJB as described above. If $\mathcal{C}^*$ is admissible, then $\mathcal{C}^*$ is an optimal contract with cost $J_0(\mathcal{C}^*) = \hat{v}(\hat{c}_l)((1-\gamma)u_0)^{\frac{1}{1-\gamma}}$.*

PROOF: The proof is very similar to Theorem 3, except we use the more general Lemma O.4 and Theorem O.2. *Q.E.D.*

The following lemma is useful to ensure the existence of an optimal contract.

LEMMA O.5: *When $\gamma \ge 1/2$, if $\alpha \le \frac{\phi\sigma\sqrt{\gamma}}{\sqrt{2}}\sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2}$ then $\hat{v} \ge \hat{c}_h^\gamma/2 > 0$. When $\gamma \le 1/2$, if $\alpha \le \phi\sigma\gamma\sqrt{2(1-\gamma)}\sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2}$ then $\hat{v} \ge (1-\gamma)(2\gamma)^{\frac{\gamma}{1-\gamma}}\hat{c}_h^\gamma > 0$.*

PROOF: For the case $\gamma \ge 1/2$, we will show that $\hat{v}_l = \hat{c}_h^\gamma/2$ is a lower bound on the cost function. To do this, it is sufficient to show that $A(\hat{c}, \hat{v}_l) \ge 0$ for any $\hat{c} \in (0, \hat{c}_h)$,

$$\begin{aligned} A(\hat{c}, \hat{v}_l) &= \hat{c} + \hat{v}_l \frac{\gamma\hat{c}_h^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}\frac{\hat{c}^{2\gamma}\left(\dfrac{\alpha}{\phi\sigma}\right)^2}{\hat{v}_l\gamma^2} \\ &\ge \hat{c} + \frac{\hat{c}_h^\gamma}{2}\frac{\gamma\hat{c}_h^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2}\hat{c}^{2\gamma}\hat{c}_h^{1-2\gamma} = \frac{\hat{c}_h^{1+\gamma}\hat{c}^{-\gamma}}{2}\left(2y^{1+\gamma} + \frac{\gamma y^\gamma - y}{1-\gamma} - y^{3\gamma}\right), \end{aligned}$$

where $y = \hat{c}/\hat{c}_h \in (0, 1)$. Since $y^{3\gamma} < y^{1+\gamma}$ the expression in parenthesis is greater or equal to

$$y^{1+\gamma} + \frac{\gamma y^{\gamma} - y}{1-\gamma}.$$

Here, we have three powers of y, and the middle coefficient is always negative, while the outside coefficients are positive (this is true both if $\gamma < 1$ and $\gamma > 1$). Moreover, the sum of the coefficients is 0 and the weighted sum (with weights equal to the powers) is

$$1 + \gamma + \frac{\gamma^2 - 1}{1-\gamma} = 0.$$

Hence, by Jensen's inequality, the expression is positive.

For the case $\gamma \le 1/2$, we will show that $\hat{v}_l = (1-\gamma)\hat{c}_m^{\gamma}$ is a lower bound, where $\hat{c}_m$ is defined by $2\gamma \hat{c}_h^{1-\gamma} = \hat{c}_m^{1-\gamma}$. We have

$$\begin{aligned}
A(\hat{c}, \hat{v}_l) &= \hat{c} + \hat{v}_l \frac{\gamma \hat{c}_h^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} \frac{\hat{c}^{2\gamma}\left(\dfrac{\alpha}{\phi\sigma}\right)^2}{\hat{v}_l \gamma^2} \\
&\ge \hat{c} + \hat{c}_m^{\gamma}\left(\hat{c}_m^{1-\gamma}/2 - \hat{c}^{1-\gamma}\right) - \frac{1}{2}\frac{\hat{c}^{2\gamma}\hat{c}_m^{1-\gamma}}{\hat{c}_m^{\gamma}} \\
&= \hat{c} + \hat{c}_m/2 - \hat{c}^{1-\gamma}\hat{c}_m^{\gamma} - \frac{1}{2}\hat{c}^{2\gamma}\hat{c}_m^{1-2\gamma} \\
&= \left(\frac{\hat{c}_m}{2}\left(1 + (\hat{c}/\hat{c}_m)^{\gamma}\right) - \hat{c}^{1-\gamma}\hat{c}_m^{\gamma}\right)\left(1 - (\hat{c}/\hat{c}_m)^{\gamma}\right).
\end{aligned}$$

If $\hat{c}/\hat{c}_m < 1$, then $\hat{c}^{1-\gamma}\hat{c}_m^{\gamma} < \hat{c}^{\gamma}\hat{c}_m^{1-\gamma} < \hat{c}_m$ and, therefore, the expression is positive. If $\hat{c}/\hat{c}_m > 1$, then $\hat{c}^{1-\gamma}\hat{c}_m^{\gamma} > \hat{c}^{\gamma}\hat{c}_m^{1-\gamma} > \hat{c}_m$ and also the expression is positive. This completes the proof. *Q.E.D.*

O.7. *Benchmark Contracts and Autarky Limit*

We can extend the benchmark contracts in Section 3 to incorporate aggregate risk. In addition, we can find conditions under which the gains from trade are exhausted and the optimal contract coincides with autarky, as mentioned in Section 4 in the paper.

Without Hidden Savings

The optimal contract without hidden savings is characterized by the HJB equation:

$$r\hat{v}_n = \min_{\sigma^x, \hat{c}, \tilde{\sigma}^x} \hat{c} - \sigma^x \hat{c}^{\gamma}\frac{\alpha}{\phi\sigma} + \hat{v}_n\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}\left(\sigma^x\right)^2 + \frac{\gamma}{2}\left(\tilde{\sigma}^x\right)^2 - \tilde{\sigma}^x\pi\right), \tag{O.22}$$

where $v_n(x) = \hat{v}_n x$ is the principal's cost function. The FOC are

$$\sigma^x = \frac{\alpha}{\gamma\left(\hat{v}_n \hat{c}_n^{-\gamma}\phi\right)\sigma}, \tag{O.23}$$

$$1 = \hat{v}_n \hat{c}^{-\gamma} + \hat{v}_n \gamma^2 \left(\sigma^x\right)^2 \hat{c}^{-1}, \tag{O.24}$$

$$\tilde{\sigma}^x = \frac{\pi}{\gamma}. \tag{O.25}$$

The optimal contract exists only if $\gamma \leq 1/2$ and only if α is sufficiently low; otherwise, the principal's value function becomes infinite.

The inverse Euler equation says that $e^{(r-\rho)t} c_t^\gamma$ is a Q-martingale. If the contract has constant $\hat{c}$, it requires

$$\sigma^x = \sqrt{\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} \frac{2}{1-2\gamma}}, \tag{O.26}$$

where

$$\hat{c}_u = \left(\frac{\rho - r(1-\gamma)}{\gamma} - (1-\gamma)\frac{1}{2}\left(\frac{\pi}{\gamma}\right)^2 \right)^{\frac{1}{1-\gamma}} \tag{O.27}$$

coincides with $\hat{c}_h$ without hidden investment.

LEMMA O.6: *The optimal contract without hidden savings satisfies the inverse Euler equation, that is, $e^{(r-\rho)t} c_t^\gamma$ is a Q-martingale, and myopic optimization over σ^x, that is, (O.23). The marginal cost of utility is lower than the inverse of the marginal utility of consumption, $\hat{v}_n < \hat{c}_n^\gamma$.*

PROOF: Myopic optimization follows from the FOC (O.23), and $\hat{v}_n < \hat{c}_n^\gamma$ from FOC (O.24). Given stationarity, the inverse Euler equation is equivalent to

$$\mu^x = \frac{r-\rho}{\gamma} + (1-\gamma)\frac{1}{2}\left(\sigma^x\right)^2 + (1+\gamma)\frac{1}{2}\left(\tilde{\sigma}^x\right)^2.$$

Using the FOC for $\hat{c}$, we can write

$$\implies \quad \hat{v}\hat{c}^{1-\gamma} = \hat{c} - \hat{v}\gamma^2\left(\sigma^x\right)^2. \tag{O.28}$$

Plug into the HJB (O.22) along with the FOC for σ^x, (O.23), to obtain

$$r\hat{v} = \hat{c} - \left(\sigma^x\right)^2 \hat{v}\gamma + \frac{1}{2}\left(\sigma^x\right)^2 \hat{v}\gamma + \hat{v}\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \hat{v}\gamma\frac{\left(\tilde{\sigma}^x\right)^2}{2}.$$

Divide by $\hat{v}$ and use (O.28), to obtain

$$\hat{c}^{1-\gamma} = \left(\frac{\rho - r(1-\gamma)}{\gamma} - (1-\gamma)\frac{\left(\tilde{\sigma}^x\right)^2}{2} \right) - \frac{\left(\sigma^x\right)^2}{2}(1-2\gamma)(1-\gamma). \tag{O.29}$$

And now compute μ^x:

$$\mu^x = \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}\left(\sigma^x\right)^2 + \frac{\gamma}{2}\left(\tilde{\sigma}^x\right)^2.$$

After some algebra, we obtain the inverse Euler equation

$$\mu^x = \frac{r-\rho}{\gamma} + (1-\gamma)\frac{1}{2}(\sigma^x)^2 + (1+\gamma)\frac{1}{2}(\tilde{\sigma}^x)^2. \qquad \textit{Q.E.D.}$$

Stationary Contracts and the Myopic Contract

Stationary contracts have a constant $\hat{c}$ and are obtained by setting $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$, $\tilde{\sigma}^x = \pi/\gamma$, and σ^x to satisfy

$$\sigma^x = \sigma_s^x(\hat{c}) \equiv \sqrt{2}\sqrt{\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}}, \tag{O.30}$$

so that $\mu^{\hat{c}} = 0$ in (17). To ensure admissibility, we must restrict

$$\hat{c} > \hat{c}_a \equiv \hat{c}_u\left(\frac{2\gamma}{1+\gamma}\right)^{\frac{1}{1-\gamma}} < \hat{c}_u, \tag{O.31}$$

so that $\mu^x - \frac{\pi^2}{\gamma} < r$. Theorem O.1 is general enough to ensure that stationary contracts are globally incentive compatible. The HJB equation (21) yields the cost of the stationary contract,

$$\hat{v}_s(\hat{c}) = \frac{\hat{c} - \dfrac{\alpha}{\phi\sigma}\hat{c}^{\gamma}\sigma_s^x(\hat{c})}{2r - \rho - (1+\gamma)\dfrac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \gamma(\pi/\gamma)^2}. \tag{O.32}$$

A special stationary contract corresponds to myopic optimization,

$$\sigma_s^x(\hat{c}_p) = \frac{\alpha}{\gamma\big(\hat{v}_s(\hat{c}_p)\hat{c}_p^{-\gamma}\phi\big)\sigma},$$

which yields

$$\begin{aligned}
&\hat{c}_p = \left(\hat{c}_h^{1-\gamma} - (1-\gamma)\frac{1}{2}\left(\frac{\alpha}{\gamma\phi\sigma}\right)^2 - (1-\gamma)\frac{1}{2}(\pi/\gamma)^2\right)^{\frac{1}{1-\gamma}}, \\
&\sigma_p^x = \frac{\alpha}{\gamma\phi\sigma}, \qquad \hat{v}_p = \hat{c}_p^{\gamma}.
\end{aligned} \tag{O.33}$$

The best stationary contract minimizes the cost, $\hat{c}_r \equiv \arg\min_{\hat{c}\in(\hat{c}_a,\hat{c}_h]} \hat{v}_s(\hat{c})$ and $\hat{v}_r \equiv \hat{v}_s(\hat{c}_r)$.

LEMMA O.7: *For any $\hat{c} \in (\hat{c}_a, \hat{c}_h]$, the corresponding stationary contract is globally incentive compatible and has cost $\hat{v}_s(\hat{c})$ given by* (O.32). *Since stationary contracts are incentive compatible, we have $\hat{v}(\hat{c}) \le \hat{v}_s(\hat{c})$.*

The myopic stationary contract is an incentive compatible stationary contract corresponding to $\hat{c}_p$, and the marginal cost of utility is equal to the inverse of the marginal utility of consumption, $\hat{v}_s(\hat{c}_p) = \hat{c}_p^{\gamma}$. The best stationary contract is less risky for the agent, that is, we have $\hat{c}_a < \hat{c}_p < \hat{c}_r$ and $\sigma_s^x(\hat{c}_r) < \sigma_p^x$. For all $\hat{c} \in (\hat{c}_p, \hat{c}_h)$, the marginal cost of utility is below the inverse of the marginal utility of consumption $\hat{v}_s(\hat{c}) < \hat{c}^{\gamma}$, and we depart from myopic optimization, $\sigma_s^x(\hat{c}) < \frac{\alpha}{\gamma(\hat{v}_s(\hat{c})\hat{c}^{-\gamma}\phi)\sigma}$.

PROOF: First, using $\alpha < \bar{\alpha}$, we can verify that $0 \le \hat{c}_h \le \hat{c}_u$, regardless of whether the agent can invest in his hidden savings. Second, $\hat{v}_s(\hat{c}) > 0$ for all $\hat{c} \in (\hat{c}_a, \hat{c}_h)$ from Lemma O.8. The same argument as in Theorem O.3 shows that $\hat{v}_s(\hat{c})$ from (O.32) is the cost corresponding to the stationary contract with $\hat{c}$ and σ^x given by (O.30), as long as the contract is indeed admissible and delivers utility u_0 to the agent. We can check that $\mu^x < r + \frac{\pi^2}{\gamma}$ for the stationary contract if and only if $\hat{c} > \hat{c}_a$. In this case, we can use Lemma O.4 to show that the stationary contract is admissible and delivers utility u_0 to the agent if $\hat{c} > \hat{c}_a$. Since the contract satisfies (O.10), (O.11), and (O.12), and (O.14) by construction, we only need to check that (O.13) holds, too. It is easy to see this is the case because $\hat{c} \le \hat{c}_h$. Theorem O.1 then ensures that it is incentive compatible.

The myopic stationary contract has $\hat{c} = \hat{c}_p$ given by (O.33). Lemma O.16 ensures that $\hat{c}_p \in (\hat{c}_a, \hat{c}_h]$ and, therefore, by the argument above, it is an incentive compatible contract. The best stationary contract has $\hat{c}_r > \hat{c}_p > \hat{c}_a$ from part 1) of Lemma O.16. From (O.30), it follows that $\sigma_p^x > \sigma_s^x(\hat{c}_r)$. Part 2) of Lemma O.16 shows that $\hat{v}_s(\hat{c}) < \hat{c}^\gamma$ for all $\hat{c} \in (\hat{c}_p, \hat{c}_h)$, with equality at $\hat{c}_p$ and $\hat{c}_h$. Therefore,

$$\sigma_s^x(\hat{c}) < \sigma_p^x = \frac{\alpha}{\gamma\phi\sigma} < \frac{\alpha}{\gamma\big(\hat{v}_s(\hat{c})\hat{c}^{-\gamma}\phi\big)\sigma}. \qquad Q.E.D.$$

LEMMA O.8: *The cost function of stationary contracts $\hat{v}_s(\hat{c})$ defined by* (O.32) *is strictly positive for all $\hat{c} \in (\hat{c}_a, \hat{c}_h]$ if and only if $\alpha < \bar{\alpha}$.*

PROOF: We need to check the numerator in (O.32), since the denominator is positive for all $\hat{c} \ge \hat{c}_a$:

$$\hat{c}\left(1 - \frac{\alpha}{\phi\sigma}\sqrt{2}\sqrt{\hat{c}^{\gamma-1}\frac{\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)\hat{c}^{\gamma-1} - 1}{1-\gamma}}\right).$$

The rest of the proof consists of evaluating this expression at $\hat{c} = \hat{c}_a$ and showing it is non-positive iff the bound is violated, since the expression is increasing in $\hat{c}$. We get $\hat{c}$ times

$$1 - \frac{\alpha}{\phi\sigma}\sqrt{2}\sqrt{1+\gamma}\frac{1}{2\gamma}\sqrt{\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{-1}}.$$

So if $\alpha \ge \bar{\alpha}$ the numerator is nonpositive, and if $\alpha < \bar{\alpha}$ then it is strictly positive. This completes the proof. *Q.E.D.*

LEMMA O.9: *If the agent has access to hidden investment, $H = \mathbb{R}_+$ and $\phi = 1$, the optimal contract is the myopic stationary contract characterized in* (O.33).

PROOF: The myopic stationary contract is both admissible and incentive compatible by Lemma O.7. Since in this case

$$\hat{c}_h = \hat{c}_p = \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\alpha}{\gamma\sigma}\right)^2 - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right),$$

we can use the same verification argument as in Theorem O.3, using the flat value function $\hat{v}(\hat{c}) = \hat{v}_p$ for all $\hat{c} \in (0, \hat{c}_h)$. For the argument to go through, it must be the case that the HJB holds as an inequality for all $\hat{c} < \hat{c}_p$:

$$A(\hat{c}, \hat{v}_p) = \hat{c} - r\hat{v}_p - \frac{1}{2}\frac{\left(\dfrac{\hat{c}^\gamma \alpha}{\phi\sigma}\right)}{\hat{v}_p\gamma} + \hat{v}_p\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma}\right) > 0.$$

This is true because $\hat{v}_p = \hat{c}_p^\gamma$, and from Lemma O.16 we know that $\partial_1 A(\hat{c}_p, \hat{c}_p^\gamma) < 0$. From Lemma O.13, we know that $A(\hat{c}, \hat{v})$ is positive near 0 and either has one root in $\hat{c}$ if $\gamma \geq 1/2$, or is convex with at most two roots if $\hat{c} \leq 1/2$. This means that $A(\hat{c}, \hat{c}_p^\gamma) > 0$ for all $\hat{c} \in (0, \hat{c}_p)$. *Q.E.D.*

O.8. *Renegotiation*

Here, we provide technical details for Section 5 of the paper. This section is consistent with the presence of aggregate risk and hidden investment introduced in Section 4 and the Online Appendix.

We say that an incentive compatible contract $\mathcal{C} = (c, k)$ is *renegotiation-proof (RP)* if

$$\infty \in \arg\min_\tau \mathbb{E}^Q\left[\int_0^\tau e^{-rt}(c - k_t\alpha)\,dt + e^{-r\tau}x_\tau \underline{\hat{v}}\right],$$

where $\underline{\hat{v}} = \inf \hat{v}(\omega, t)$. The optimal contract with hidden savings is not renegotiation proof, because after any history $\hat{v}_t > \hat{v}_l = \underline{\hat{v}}$, so the principal is always tempted to "start over." In fact, it is easy to see that RP contracts must have a constant $\hat{v}_t$. The converse it also true.

LEMMA O.10: *An incentive compatible contract $\mathcal{C}$ is renegotiation proof if and only if the continuation cost $\hat{v}$ is constant.*

PROOF: If ever $\hat{v}_t > \underline{\hat{v}}$, then renegotiating at that point is better than never renegotiating and obtaining $\hat{v}_0$. In the other direction, if $\hat{v}$ is constant, any stopping time τ yields the same value to the principal, so $\tau = \infty$ is an optimal choice. *Q.E.D.*

Stationary contracts have a constant $\hat{v}$, because $\hat{c}$ is constant. However, those contracts were built using $dL_t = 0$. There are other contracts with a constant $\hat{c}$ that use $dL_t > 0$, that is, the drift of $\hat{c}$ would be negative without dL_t. In addition, there could be nonstationary contracts with a constant cost $\hat{v}(\hat{c})$ for all $\hat{c}$ in the domain. The next lemma shows they are all worse than the best stationary contract $\mathcal{C}_r$, with cost $\hat{v}_r = \min_{\hat{c} \in (\hat{c}_a, \hat{c}_h]} \hat{v}_s(\hat{c})$.

THEOREM O.4: *The optimal renegotiation-proof contract is the optimal stationary contract $\mathcal{C}_r$ with cost $\hat{v}_r$.*

PROOF: Since the optimal stationary contract is incentive compatible and has a constant $\hat{v}$, we only need to show that any incentive compatible contract with constant $\hat{v}$ has $\hat{v} \geq \hat{v}_r$. This is clearly true for all stationary contracts as defined in Lemma O.7 with aggregate risk.

There could also be stationary contracts with a constant $\hat{c}$ but $dL_t > 0$. For these contracts, the drift $\mu^{\hat{c}} < 0$ in the absence of dL_t. Consider the optimization problem

$$0 = \min_{\sigma^x} \hat{c} - r\hat{v} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} + \hat{v}\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)$$

$$\text{s.t.} \quad \frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{1}{2}(\sigma^x)^2 + \frac{1}{2}\left(\frac{\pi}{\gamma}\right)^2 \leq 0.$$

If the constraint is binding, we get the stationary contracts with $dL_t = 0$, so $\hat{v} = \hat{v}_s$. We want to show that it must be binding. Toward contradiction, if the constraint is not binding we have $\sigma^x = \frac{\alpha}{\sigma\gamma}\frac{\hat{c}_l^\gamma}{\hat{v}_l\phi}$ and, therefore, we have $A(\hat{c}, \hat{v}) = 0$, where A is defined as in Lemma O.13. If $\hat{v} \leq \hat{v}_r$, then $\hat{v} \leq \hat{v}_p$, because the myopic stationary contract is incentive compatible ($\hat{c}_p \leq \hat{c}_h$ for any valid hidden investment). Then Lemma O.15 ensures that $\frac{r-\rho}{\gamma} - \frac{\rho-\hat{c}^{1-\gamma}}{1-\gamma} + \frac{1}{2}(\sigma^x)^2 + \frac{1}{2}(\frac{\pi}{\gamma})^2 > 0$, which violates the constraint. This means that $\hat{v} \geq \hat{v}_r$.

Finally, if we have a nonstationary contract with a constant $\hat{v} < \hat{v}_r$, the domain of $\hat{c}$ must have an upper bound $\bar{c} \leq \hat{c}_h$ because otherwise they would have a lower cost than the optimal contract near $\hat{c}_h$, and this cannot be for an IC contract. For the upper bound $\bar{c}$, we must have $\sigma^{\hat{c}} = 0$ and $\mu^{\hat{c}} \leq 0$. But this is the same situation with stationary contracts with $dL_t > 0$, and we know their cost is above $\hat{v}_r$. *Q.E.D.*

REMARK: It is possible that $\hat{c}_r = \hat{c}_h$ if the agent can invest his hidden savings and ϕ is close enough to 1. In the special case with hidden investment and $\phi = 1$, we have $\hat{c}_r = \hat{c}_p = \hat{c}_h$, as shown in Lemma O.9.

O.9. *Intermediate Results*

LEMMA O.11: *The cost function $\hat{v}$ is flat on $(0, \hat{c}_l)$, $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_l)$, and the HJB equation holds as an inequality in that region, $A(\hat{c}, \hat{v}(\hat{c})) \geq 0$. For $\hat{c} \in (\hat{c}_l, \hat{c}_h)$, the cost function is C^2, strictly increasing with $\hat{v}'(\hat{c}) > 0$, and satisfies the HJB equation. At $\hat{c}_l$, we have the smooth pasting condition $\hat{v}'(\hat{c}_l) = 0$.*

PROOF: Denote $\hat{v}$ the true cost function. We will use f to denote test functions, and sometimes use f, f', and f'' to denote its value and derivatives at a point $\hat{c}$. Because the $dL_t \geq 0$ term in the law of motion of $\hat{c}_t$ allows it to go up at any time, $\hat{v}$ must be nondecreasing and it can have a flat region $(0, \hat{c}_l)$ where $\hat{c}_t$ would jump up to $\hat{c}_l$, so $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_l)$ for all $\hat{c} \in (0, \hat{c}_l)$. $\hat{c}_l > 0$ because $\hat{c} = 0$ requires not giving the agent any capital, so it is just delaying the start of the contract, which is not optimal because $\rho > r(1-\gamma)$.

Recall the HJB equation, for a generic test function f,

$$\begin{aligned} 0 = \min_{\sigma^x \geq 0, \sigma^{\hat{c}}} & \hat{c} - rf - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma}\right) \\ & + f'\hat{c}\left(\frac{\hat{c}^{1-\gamma} - (\hat{c}_u)^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2} + (1+\gamma)\sigma^x\sigma^{\hat{c}} + \frac{1+\gamma}{2}(\sigma^{\hat{c}})^2\right) \\ & + \frac{f''}{2}\hat{c}^2\left((\sigma^{\hat{c}})^2\right). \end{aligned} \tag{O.34}$$

If the agent has access to hidden investment, the minimization is subject to the constraint $\sigma^x + \sigma^{\hat{c}} \geq \alpha/(\sigma\gamma)$. Notice we already plugged in $\tilde{\sigma}^x = \pi/\gamma$ and $\tilde{\sigma}^{\hat{c}} = 0$. Recall $\hat{c}_u$ is defined in (O.27).

Before going into the proof, let us review a few facts. A C^2 function f is called a supersolution of (O.34) if instead of equality, it satisfies the inequality

$$0 \geq \min_{\sigma^x, \sigma^{\hat{c}}} \ldots$$

For a supersolution f, if it is possible to attain points $\hat{c}^-$ and $\hat{c}^+$ at cost less than or equal to $f(\hat{c}^-)$ and $f(\hat{c}^+)$, respectively, then the contract that satisfies the inequality above attains any point $\hat{c} \in [\hat{c}^-, \hat{c}^+]$ with a cost of less or equal to $f(\hat{c})$, as long as the contract is admissible.

A C^2 subsolution satisfies

$$0 \leq \min_{\sigma^x, \sigma^{\hat{c}}} \ldots$$

If the cost of attaining points $\hat{c}^-$ and $\hat{c}^+$ is greater than or equal to $f(\hat{c}^-)$ and $f(\hat{c}^+)$, then the cost of attaining $\hat{c} \in [\hat{c}^-, \hat{c}^+]$ is greater than $f(\hat{c})$. We call functions *strict* super and subsolutions if the corresponding inequality is strict. If f is locally a strict supersolution, then a perturbation of f, for example, a small translation or rotation, is also locally a strict supersolution (and a similar statement holds for strict subsolutions). We will use super and subsolutions as test functions around the true cost $\hat{v}$ to prove properties of $\hat{v}$ (such as differentiability).

Next fact, equation (O.34) implies a value of f'' only for some triples $(\hat{c}, f, f')$. Let us elaborate. Let us write the HJB equation for deterministic contracts, in which we must choose $\sigma^{\hat{c}} = 0$, as

$$\hat{A}(\hat{c}, f, f') = 0, \tag{O.35}$$

where

$$\hat{A}(\hat{c}, f, f') \equiv \min_{\sigma^x \geq 0} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma}\right) - rf$$
$$+ f'\hat{c}\left(\frac{\hat{c}^{1-\gamma} - (\hat{c}_u)^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2}\right),$$

subject to $\sigma^x \geq \alpha/(\phi\sigma)$ if the agent has access to hidden investment.

Notice that $\hat{A}(\hat{c}, f, f')$ is concave in f' as the minimum of linear functions, and that $\hat{A}$ goes to $-\infty$ as f' goes to ∞ or $-\gamma f/\hat{c}$. For $f' \in (-\gamma f/\hat{c}, \infty)$, the optimal choice of σ^x is

$$\sigma^x = \hat{c}^\gamma \frac{\alpha}{\phi\sigma} \frac{1}{\gamma f + f'\hat{c}}$$

if the hidden investment constraint is not binding, and when this leads to

$$\frac{(\sigma^x)^2}{2} = \frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma},$$

the hidden investment constraint is not binding (because $\hat{c} \le \hat{c}_h$) and function $\hat{A}(\hat{c}, f, f')$ achieves its maximum in variable f', because then

$$\hat{A}_3(\hat{c}, f, f') = 0.$$

Thus, we have

$$\max_{f'} \hat{A}(\hat{c}, f, f') = \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma}\right) - rf,$$
$$\sigma^x = \sqrt{2\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}}.$$

When $\max_{f'} \hat{A}(\hat{c}, f, f') > 0$, then the equation $\hat{A}(\hat{c}, f, f') = 0$ has two roots $\tilde{f}'_L$ and $\tilde{f}'_R > \tilde{f}'_L$, in the range $f' = (-\gamma f/\hat{c}, \infty)$. Then at point $(\hat{c}, f)$ it is possible to solve the deterministic equation as a first-order ODE with slopes $f'(\hat{c}) = \tilde{f}'_L$ and $\tilde{f}'_R$. The former solution has positive drift at $\hat{c}$, points on the solution to the left of $\hat{c}$ are attainable if $(\hat{c}, f)$ is attainable. The latter solution has negative drift at $\hat{c}$, and points on the solution to the right of $\hat{c}$ are attainable if $(\hat{c}, f)$ is attainable. When $\max_{f'} \hat{A}(\hat{c}, f, f') = 0$, we can say that $\tilde{f}'_L = \tilde{f}'_R$ is the unique root.

In fact, $\max_{f'} \hat{A}(\hat{c}, f, f') \ge 0$ if and only if $f < \hat{v}_s(\hat{c})$ on $(\hat{c}_a, \hat{c}_h)$ and if and only if $f > \hat{v}_s(\hat{c})$ on $(0, \hat{c}_a)$. Recall that $\hat{c}_a$ is defined in (O.31), and recall that the curve $f = \hat{v}_s(\hat{c})$ is defined by

$$\hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma}\right) - rf = 0,$$
$$\sigma^x = \sqrt{2\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}}.$$

For $\hat{c} \in (\hat{c}_a, \hat{c}_h)$, we have $\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma} < r$, hence $\max_{f'} \hat{A}(\hat{c}, f, f') \ge 0$ if and only if $f < \hat{v}_s(\hat{c})$. For $\hat{c} < \hat{c}_a$, $\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma} > r$, hence $\max_{f'} \hat{A}(\hat{c}, f, f') \ge 0$ if and only if $f > \hat{v}_s(\hat{c})$. We have $\max_{f'} \hat{A}(\hat{c}, f, f') = 0$ if and only if $f = \hat{v}_s(\hat{c})$.

Whenever $\max_{f'} \hat{A}(\hat{c}, f, f') > 0$, if $f' \in (\tilde{f}'_L, \tilde{f}'_R)$ and $f' > 0$, equation (O.34) implies a unique value of f'', and can be solved locally as a second-order ordinary differential equation (ODE). To see this, notice that with $f'' = \infty$, the right-hand side becomes equal to $\hat{A}(\hat{c}, f, f') > 0$, and is increasing in f''. As $f'' \to -\gamma(f - f'\hat{c})\hat{c}^{-1}\frac{(1+\gamma)f'}{f\gamma + f'\hat{c}}$ from above, the objective function diverges to $-\infty$ if the hidden investment constraint is not binding as we approach the limit. If the hidden investment constraint is binding, then as $f'' \to -\gamma(f - f'\hat{c})\hat{c}^{-1}$ the objective function diverges to $-\infty$ if $\gamma(f - f'\hat{c})\hat{c}\frac{\alpha}{\sigma\gamma} < \hat{c}^\gamma \frac{\alpha}{\phi\sigma}$; if not, then it means we hit the $\sigma^x \ge 0$ constraint and we must set $\sigma^{\hat{c}} = \alpha/(\sigma\gamma)$ and the objective function diverges to $-\infty$ as $f'' \to -\infty$. So we have a unique f'' that solves (O.34). The equation is locally Lipschitz-continuous, as long as we stay in the region where $\max_{f'} \hat{A}(\hat{c}, \hat{v}(\hat{c}), f') > 0$, which implies all the usual properties (existence, uniqueness, and continuity in initial conditions).

Notice that also if $\max_{f'} \hat{A}(\hat{c}, f, f') \leq 0$, and $f' > 0$, then any C^2 function is locally a strict supersolution, no matter how high f'' is. Indeed, if we set $\sigma^{\hat{c}} = 0$ and minimize over σ^x, we get the inequality $\geq$ in equation (O.34), and we can make the inequality strict using $\sigma^{\hat{c}}$. Also, for any triple $(\hat{c}, f, f')$ with $f' > 0$, a sufficiently concave C^2 function is locally a strict supersolution.

Now, let us prove some regularity properties of function $\hat{v}$. First, left and right derivatives $\hat{v}'_-(\hat{c})$ and $\hat{v}'_+(\hat{c})$ exist. If not, for example, if

$$\liminf_{\hat{c}_n \to \hat{c}} \frac{\hat{v}(\hat{c}) - \hat{v}(\hat{c}_n)}{\hat{c} - \hat{c}_n} < \limsup_{\hat{c}_n \to \hat{c}} \frac{\hat{v}(\hat{c}) - \hat{v}(\hat{c}_n)}{\hat{c} - \hat{c}_n}$$

for some sequence $\{\hat{c}_n\}$ converging to $\hat{c}$ from below, then we can take a local strict supersolution f with $f(\hat{c}) = \hat{v}(\hat{c})$ and $f'(\hat{c})$ between these two bounds. Points $(\hat{c}_n, \hat{v}(\hat{c}_n))$ above f for sufficiently large n can be improved upon by a contract based on the solution f.

Second, we have $\hat{v}'_-(\hat{c}) \leq \hat{v}'_+(\hat{c})$. If not, that is, $\hat{v}'_-(\hat{c}) > \hat{v}'_+(\hat{c})$, then a local supersolution f with $f(\hat{c}) = \hat{v}(\hat{c})$ and $f'(\hat{c}) = (\hat{v}'_-(\hat{c}) + \hat{v}'_+(\hat{c}))/2$ can be used to improve upon the optimal contract with value $\hat{v}(\hat{c})$. Indeed, if we slightly lower $f(\hat{c}) = \hat{v}(\hat{c}) - \epsilon$, the solution is still a local strict supersolution that goes above $\hat{v}$ on both sides of $\hat{c}$, and the corresponding contract has cost less than or equal to $\hat{v}(\hat{c}) - \epsilon$ at $\hat{c}$.

Third, we have $\max_{f'} \hat{A}(\hat{c}, \hat{v}(\hat{c}), f') \geq 0$. For $\hat{c} \in (\hat{c}_a, \hat{c}_h)$, this follows immediately because $\hat{v}(\hat{c}) \leq \hat{v}_s(\hat{c})$, because stationary contracts provide an upper bound on the cost function from the optimal contract. For $\hat{c} \in (0, \hat{c}_a)$, the argument is a bit more involved. Consider the time-varying version of the HJB equations with choices $\sigma^{\hat{c}} = 0$ and

$$\sigma^x = \sqrt{2\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}},$$

which satisfies the hidden investment constraint for all $\hat{c} \leq \hat{c}_h$,

$$\frac{\partial f}{\partial t} + \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma}\right) - rf = 0.$$

For $\hat{c} \in (0, \hat{c}_a)$ and $f < \hat{v}_s(\hat{c})$, this equation implies $\partial f \partial t > 0$, that is, this choice of controls leads to f drifting straight up. This means that if $\hat{v}(\hat{c}) < \hat{v}_s(\hat{c})$ on $(0, \hat{c}_a)$, this contract allows us to achieve lower cost.

Fourth, let us show that $\max_{f'} \hat{A}(\hat{c}, \hat{v}(\hat{c}), f') > 0$ everywhere, that is, $\hat{v}(\hat{c}) \neq \hat{v}_s(\hat{c})$. At any point $\hat{c} \in (\hat{c}_a, \hat{c}_h)$, when $\hat{v}'_s(\hat{c}) < 0$ then the principal can get a better value than $\hat{v}_s(\hat{c})$ by switching to the optimal stationary contract slightly above $\hat{c}$. At any point $\hat{c} \in (0, \hat{c}_a)$, when $\hat{v}'_s(\hat{c}) < 0$, if it were the case that $\hat{v}(\hat{c}) = \hat{v}_s(\hat{c})$, then the principal could achieve $\hat{v}(\hat{c})$ at $\hat{c} - \epsilon$, so $\hat{v}(\hat{c} - \epsilon) \leq \hat{v}(\hat{c}) < \hat{v}_s(\hat{c} - \epsilon)$, which we know cannot be. When $\hat{c} \in (\hat{c}_a, \hat{c}_h)$ and $\hat{v}'_s(\hat{c}) > 0$, we can conclude that $\hat{v}(\hat{c}) < \hat{v}_s(\hat{c})$ by the following argument. Any C^2 function which satisfies $f(\hat{c}) = \hat{v}_s(\hat{c})$, $f'(\hat{c}) = \hat{v}'_s(\hat{c}) > 0$, including those that go above $\hat{v}_s$ in the neighborhood of $\hat{c}$, is locally a strict supersolution. Hence, $\hat{v}_s(\hat{c}) - \epsilon$ is locally attainable for sufficiently small ϵ.

The following lemma is helpful to deal with the remaining cases (recall $\hat{c}_p$ is defined in (O.33) as the myopic stationary contract).

LEMMA O.12: *When $A(\hat{c}, f) \leq 0$ and $f < \hat{c}^\gamma$, then $\hat{A}_3(\hat{c}, f, 0) > 0$. Hence, for any $\hat{c} \in (\hat{c}_p, \hat{c}_h)$ and any $\hat{c} \in (0, \hat{c}_a)$, at $(\hat{c}, \hat{v}_s(\hat{c}))$, $\tilde{f}'_L = \tilde{f}'_R > 0$.*

PROOF: We have

$$\hat{A}_3(\hat{c}, f, 0) = \left(\frac{\hat{c}^{1-\gamma} - (\hat{c}_u)^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2}\right).$$

Since $A(\hat{c}, f) \leq 0$, it means that

$$\hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} - rf + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right) \leq 0,$$

$$f\frac{\gamma}{2}(\sigma^x)^2 \geq \hat{c} + f\frac{\gamma(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}, \qquad \sigma^x = \hat{c}^\gamma \frac{\alpha}{\phi\sigma}\frac{1}{\gamma f}.$$

Toward a contradiction, suppose $\hat{A}_3(\hat{c}, f, 0) \leq 0$. Then

$$\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} \geq \frac{(\sigma^x)^2}{2}$$
$$\Rightarrow \quad \gamma f\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} \geq f\frac{\gamma}{2}(\sigma^x)^2 \geq \hat{c} + f\frac{\gamma(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma} \quad \Rightarrow \quad f \geq \hat{c}^\gamma,$$

a contradiction.

Now, we have $A(\hat{c}, \hat{v}_s(\hat{c})) < 0$ and $\hat{v}_s(\hat{c}) < \hat{c}^\gamma$ for any $\hat{c} \in (0, \hat{c}_a) \cup (\hat{c}_p, \hat{c}_h)$, so at those points we have $\tilde{f}'_L = \tilde{f}'_R > 0$. *Q.E.D.*

It follows from the lemma that starting from the minimum of $\hat{v}_s$ on $(\hat{c}_a, \hat{c}_h)$ to the left, we can solve the equation (O.35) with slope $\tilde{f}'_L > 0$ (locally) with nonnegative drift. This solution is attainable,[24] hence $\hat{v}$ must be at the level of this solution or below, but $0 < f'_L \leq \hat{v}'_- \leq \hat{v}'_+ \leq \hat{v}'_s(\hat{c}) = 0$, which contradicts this. So $\hat{v}$ is below $\hat{v}_s$ everywhere on $\hat{c} \in (\hat{c}_a, \hat{c}_h)$, including at the minimum of $\hat{v}_s$.

Now, let us rule out the possibility that $f = \hat{v}(\hat{c}) = \hat{v}_s(\hat{c})$ on the increasing portion of $\hat{v}_s$ (including the local maximum) in the range $(0, \hat{c}_a)$. Then $A_3(\hat{c}, f, \tilde{f}'_L = \tilde{f}'_R) = 0$. Hence, for

$$\sigma^x = \sqrt{2\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}} - \epsilon,$$

The value of

$$\underbrace{\hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\sigma} + f\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}(\sigma^x)^2 - \frac{1}{2}\frac{\pi^2}{\gamma}\right) - rf}_{>0, O(\epsilon)}$$
$$+ f'\hat{c}\underbrace{\left(\frac{\hat{c}^{1-\gamma} - (\hat{c}_u)^{1-\gamma}}{1-\gamma} + \frac{(\sigma^x)^2}{2}\right)}_{<0, O(\epsilon)} < 0,$$

[24]This solution corresponds to a deterministic contract, in which $\hat{c}_t$ converges slowly to the minimum of $\hat{v}_s$, as the drift gets closer and closer to 0.

on the order of ϵ^2. Hence, we can satisfy this equation by setting f' slightly lower than $\tilde{f}'_L$. For that choice of σ^x, we get a deterministic contract with positive drift near $\hat{c}$, and the curve that corresponds to this contract is an upper bound on $\hat{v}$. Likewise, by choosing

$$\sigma^x = \sqrt{2\frac{(\hat{c}_u)^{1-\gamma} - \hat{c}^{1-\gamma}}{1-\gamma}} + \epsilon,$$

we get a contract with slope slightly higher than $\tilde{f}'_R$. These contracts allow us to achieve values below $\hat{v}_s$ (which is impossible), unless $\tilde{f}'_L = \tilde{f}'_R = \hat{v}'_s(\hat{c})$. In the latter case, letting $\epsilon \to 0$, we find that $\hat{v}' = \tilde{f}'_L = \tilde{f}'_R$. Now, any C^2 function which satisfies $f(\hat{c}) = \hat{v}_s(\hat{c})$, $f'(\hat{c}) = \hat{v}'_s(\hat{c}) > 0$, including those that go above $\hat{v}$ in the neighborhood of $\hat{c}$, is locally a strict supersolution. Hence, $\hat{v}_s(\hat{c}) - \epsilon$ is locally attainable for sufficiently small ϵ, a contradiction. We conclude that $\max_{f'} \hat{A}(\hat{c}, \hat{v}(\hat{c}), f') > 0$ for all $\hat{c} \in (0, \hat{c}_h)$.

Now, at any $\hat{c}$, the slope $\hat{v}'_+$ cannot be steeper than $\tilde{f}'_R$, or else we can improve upon the cost function $\hat{v}$ to the right of $\hat{c}$ through a deterministic solution with slope $\tilde{f}'_R$ that passes through $(\hat{c}, \hat{v}(\hat{c}))$. Likewise, the slope $\hat{v}'_-$ cannot be less than $\tilde{f}'_L$ and cannot be negative.

We already showed that $\hat{v}'_- \leq \hat{v}'_+$. The inequality cannot be strict, or else the equation (O.34) is solvable as a second-order ODE at $\hat{c}$ with $f(\hat{c}) = \hat{v}(\hat{c}) + \epsilon$ and slope $f'(\hat{c}) = (\hat{v}'_- + \hat{v}'_+)/2 \in (\tilde{f}'_L, \tilde{f}'_R)$, for sufficiently small ϵ. Because this is a subsolution, this implies that cost $\hat{v}(\hat{c})$ at $\hat{c}$ is unattainable. To sum up, the derivative $\hat{v}'$ exists and must be in the interval $[\tilde{f}'_L, \tilde{f}'_R]$ and nonnegative.

Now, let us show that $\hat{v}' \in (\tilde{f}'_L, \tilde{f}'_R)$ whenever $\hat{v}' > 0$. Otherwise, any C^2 test function f with $f = \hat{v}$, $f' = \hat{v}'$ and arbitrarily large f'' is locally a strict supersolution. Suppose $\hat{v}' = \tilde{f}'_R$, then the solution of (O.35) with this initial condition to the right of $\hat{c}$ is weakly above $\hat{v}$ and has finite second derivative. The test function f goes strictly above the solution of (O.35) to the right of $\hat{c}$, assuming f'' is large enough. We can rotate the test function clockwise slightly, it remains a supersolution that goes below $\hat{v}$ and then above to the right of $\hat{c}$. When it goes below, those points are attainable, hence we can improve upon the cost function $\hat{v}$, a contradiction.

Since $\hat{v}' \in (\tilde{f}'_L, \tilde{f}'_R)$, if $\hat{v}' > 0$, we can solve (O.34) locally with initial conditions $(\hat{c}, \hat{v}, \hat{v}')$. If the solution f does not coincide with $\hat{v}$ locally, if it goes above, then we can rotate it slightly to find points below $\hat{v}$ that are attainable. If it goes below, then likewise we can rotate it slightly to find points above $\hat{v}$ that are unattainable. Hence, the tangent solution of (O.34) must coincide with $\hat{v}$ locally.

To sum up, whenever $\hat{v}' > 0$, the cost function $\hat{v}$ satisfies the HJB equation (O.34) as a second-order ODE.

Now, whenever also $A(\hat{c}, \hat{v}(\hat{c})) < 0$ we know that $\hat{v}(\hat{c}) < \hat{c}^\gamma$, and we can rule out the possibility that $\hat{v}' = 0$ because otherwise $\tilde{f}'_L > 0$ and we can improve upon $\hat{v}$ using the solution of the deterministic equation (O.35) with slope $\tilde{f}'_L > 0$ at $(\hat{c}, \hat{v})$ (and positive drift). To see that $A(\hat{c}, \hat{v}(\hat{c})) < 0$ implies $\hat{v}(\hat{c}) < \hat{c}^\gamma$, use Lemma O.16, and notice that $\hat{v}(\hat{c}) \geq \hat{c}^\gamma$ can only occur for $\hat{c} \leq \hat{c}_p$ because $\hat{v}(\hat{c}) < \hat{v}_s(\hat{c}) < \hat{c}^\gamma$ for $\hat{c} > \hat{c}_p$. For $\hat{c} \leq \hat{c}_p$, we know $A(\hat{c} - \delta, \hat{c}^\gamma) \geq 0$ for any $\delta \in [0, \hat{c}]$. So $A(\hat{c}, \hat{v}(\hat{c})) < 0$ implies $\hat{v}(\hat{c}) < \hat{c}^\gamma$.

We also know that if $\hat{v}'(\hat{c}) > 0$ for some $\hat{c} \in (0, \hat{c}_h)$, then $\hat{v}'(\hat{c}') > 0$ and the HJB holds for all $\hat{c}' \in (\hat{c}, \hat{c}_h)$. To see why, if $\hat{v}'(\hat{c}) > 0$ then $\hat{v}$ is C^2 and the HJB holds in a neighborhood of $\hat{c}$. We must always have

$$\hat{v}'\hat{c}(1+\gamma) + \hat{v}''\hat{c}^2 \geq 0.$$

Otherwise, we can set $\sigma^x = 0$ and $\sigma^{\hat{c}}$ arbitrarily large, satisfying the hidden investment constraint and getting an arbitrarily negative value on the left-hand side of the HJB. Rearrange to get

$$\hat{v}'' \geq -\hat{v}'\hat{c}^{-1}(1+\gamma).$$

Use Gronwall's inequality to get

$$\hat{v}'(\hat{c}') \geq \hat{v}'(\hat{c}) \times e^{\int_{\hat{c}}^{\hat{c}'} -x^{-1}(1+\gamma)\,dx} > 0 \quad \forall \hat{c}' \in (\hat{c}, \hat{c}_h),$$

and, therefore, $\hat{v}$ satisfies the HJB in $(\hat{c}, \hat{c}_h)$.

It follows that $\hat{v}'$ could be zero only in $(0, \hat{c}_l]$, where $A(\hat{c}, \hat{v}(\hat{c})) \geq 0$ and the HJB therefore holds only as an inequality, but the derivative $\hat{v}'$ must become strictly positive before $A(\hat{c}, \hat{v}(\hat{c})) < 0$, so $\hat{c}_l = \inf\{\hat{c} : \hat{v}'(\hat{c}) > 0\} \in (0, \hat{c}_h)$. Once $\hat{v}'(\hat{c}) > 0$, it remains strictly positive and the HJB holds for all $\hat{c} \in (\hat{c}_l, \hat{c}_h)$. Since we know there are no kinks, we have the smooth pasting condition $\hat{v}'(\hat{c}_l) = 0$ at $\hat{c}_l$. *Q.E.D.*

LEMMA O.13: *Define the function*

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2}\frac{\left(\dfrac{\hat{c}^\gamma \alpha}{\phi\sigma}\right)^2}{\hat{v}\gamma} + \hat{v}\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma}\right).$$

For any $\hat{v} \in (0, (\hat{c}_u)^\gamma)$, we have $A(\hat{c}; \hat{v}) > 0$ for $\hat{c}$ near 0, where $\hat{c}_u = (\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2)^{\frac{1}{1-\gamma}}$. In addition, if $\gamma \geq \frac{1}{2}$ then $A(\hat{c}; \hat{v})$ has at most one root in $[0, \hat{c}_u]$. If instead $\gamma < \frac{1}{2}$, $A(\hat{c}; \hat{v})$ is convex and has at most two roots.

PROOF: First, for $\gamma < 1$ $\lim_{\hat{c}\to 0} A(\hat{c}; \hat{v}) = \hat{v}\frac{\gamma}{1-\gamma}(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2) > 0$. For $\gamma > 1$, $\lim_{\hat{c}\to 0} A(\hat{c}; \hat{v}) = \infty$.

For $\gamma \geq 1/2$, to show that $A(\hat{c}; \hat{v})$ has at most one root in $[0, \hat{c}_u]$ for any $\hat{v} \in (0, \hat{v}_h)$, we will show that $A'_{\hat{c}}(\hat{c}; \hat{v}) = 0 \implies A(\hat{c}; \hat{v}) > 0$ for all $\hat{c} < \hat{c}_u$. Compute the derivative (dropping the arguments to avoid clutter)

$$A'_{\hat{c}} = 1 - \hat{v}\hat{c}^{-\gamma} - \hat{c}^{2\gamma-1}\left(\frac{\alpha}{\phi\sigma}\right)^2 \frac{1}{\hat{v}}.$$

So

$$A'_{\hat{c}} = 0 \quad \implies \quad \hat{c} - \hat{v}\hat{c}^{1-\gamma} = \hat{c}^{2\gamma}\left(\frac{\alpha}{\phi\sigma}\right)^2 \frac{1}{\hat{v}}.$$

Plug this into the formula for A to get

$$\begin{aligned}
A &= \hat{c} - r\hat{v} + \hat{v}\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\hat{c}^{2\gamma}}{2\hat{v}\gamma}\left(\frac{\alpha}{\phi\sigma}\right)^2 - \frac{\hat{v}}{2}\frac{\pi^2}{\gamma}, \\
A &= \hat{c} - r\hat{v} + \hat{v}\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2\gamma}\left(\hat{c} - \hat{v}\hat{c}^{1-\gamma}\right) - \frac{\hat{v}}{2}\frac{\pi^2}{\gamma} \\
&= \frac{2\gamma - 1}{2\gamma}\hat{c} + \frac{1 - 3\gamma}{2\gamma}\hat{v}\frac{\hat{c}^{1-\gamma}}{1-\gamma} + \hat{v}\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{\hat{v}}{2}\frac{\pi^2}{\gamma} \equiv B(\hat{c}, \hat{v}).
\end{aligned}$$

$B(\hat{c}, \hat{v})$ is convex in $\hat{c}$ because $1-3\gamma < 0$ for $\gamma \geq \frac{1}{2}$, so it is minimized in $\hat{c}$ when $B'_{\hat{c}} = 0$:

$$\frac{2\gamma - 1}{3\gamma - 1} = \hat{v}\hat{c}^{-\gamma}, \tag{O.36}$$

and it is strictly decreasing before this point. Now we have two possible cases.

CASE 1: The minimum of B is achieved for $\hat{c} \geq \hat{c}_u$, so in the relevant range, it is minimized at $\hat{c}_h$. So let us plug in $\hat{c}_u$ into $B(\hat{c}, \hat{v})$:

$$\begin{aligned}
2\gamma B(\hat{c}_u, \hat{v}) &= (2\gamma - 1)\hat{c}_u + \frac{\hat{v}}{1-\gamma}\big(\big(\rho - r(1-\gamma)\big)2\gamma + (1-3\gamma)(\hat{c}_u)^{1-\gamma}\big) - \hat{v}\pi^2 \\
&= (2\gamma - 1)\hat{c}_u + \frac{\hat{v}}{1-\gamma}\frac{\big(\rho - r(1-\gamma)\big)}{\gamma}\big(2\gamma^2 + (1-3\gamma)\big) \\
&\quad - \frac{\hat{v}}{1-\gamma}(1-3\gamma)\frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2 - \hat{v}\pi^2 \\
&= (2\gamma - 1)\hat{c}_u + \hat{v}\left(\frac{\big(\rho - r(1-\gamma)\big)}{\gamma}\right)(1-2\gamma) \\
&\quad - \frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2\hat{v}\left(\frac{1-3\gamma}{1-\gamma} + \frac{2\gamma^2}{1-\gamma}\right) \\
&= (2\gamma - 1)\hat{c}_u + \hat{v}\left(\frac{\big(\rho - r(1-\gamma)\big)}{\gamma}\right)(1-2\gamma) - \frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2\hat{v}(1-2\gamma) \\
&= (2\gamma - 1)\left(\hat{c}_u - \hat{v}\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2\right)\right) \geq 0,
\end{aligned}$$

and the inequality is strict if $\hat{v} < (\hat{c}_u)^\gamma$. So $\mathcal{A}(\hat{c}, \hat{v}) = B(\hat{c}, \hat{v}) > B(\hat{c}_u, \hat{v}) \geq 0$ for any $\hat{c} < \hat{c}_u$.

CASE 2: If the minimum is achieved for $\hat{c}_m \in [0, \hat{c}_u)$, it must be that $\gamma > 1/2$. Then plugging in (O.36) into B:

$$\begin{aligned}
B(\hat{c}, \hat{v}) &\geq \frac{2\gamma - 1}{2\gamma}\hat{c}_m - \frac{2\gamma - 1}{2\gamma}\frac{\hat{c}_m}{1-\gamma} + \hat{v}\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{\hat{v}}{2}\frac{\pi^2}{\gamma} \\
&= \frac{1-2\gamma}{2}\frac{\hat{c}_m}{1-\gamma} + \hat{v}\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{\hat{v}}{2}\frac{\pi^2}{\gamma} \\
&= \frac{1-2\gamma}{2}\frac{\hat{c}_m}{1-\gamma} + \frac{2\gamma - 1}{3\gamma - 1}\hat{c}_m^\gamma\left(\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma}\right),
\end{aligned}$$

and dividing throughout by $2\gamma - 1 > 0$,

$$\frac{B(\hat{c}, \hat{v})}{2\gamma - 1} \geq -\frac{1}{2}\frac{\hat{c}_m}{1-\gamma} + \frac{\hat{c}_m^\gamma}{3\gamma - 1}\left(\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma}\right),$$

and multiplying by $\hat{c}_m^{-\gamma} > 0$ and using $\frac{\hat{c}_m^{1-\gamma}}{1-\gamma} < \frac{(\hat{c}_u)^{1-\gamma}}{1-\gamma}$,

$$\begin{aligned}
\frac{B(\hat{c}, \hat{v})}{2\gamma - 1} \hat{c}_m^{-\gamma} &> -\frac{1}{2} \frac{(\hat{c}_u)^{1-\gamma}}{1-\gamma} + \frac{1}{3\gamma - 1} \left(\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right) \\
&= \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2 \right) \left(-\frac{1}{2} \frac{1}{1-\gamma} + \frac{\gamma}{(3\gamma-1)(1-\gamma)} \right) \\
&= \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2 \right) \frac{1 - 3\gamma + 2\gamma}{(1-\gamma)(3\gamma-1)2} \\
&= \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2 \right) \frac{1}{(3\gamma-1)2} > 0.
\end{aligned}$$

So $A(\hat{c}; \hat{v}) \geq B(\hat{c}, \hat{v}) > 0$ for all $\hat{c} \in [0, \hat{c}_u]$.

For the case with $\gamma < \frac{1}{2}$, the second derivative of A is

$$A''_{\hat{c}} = \gamma \hat{v} \hat{c}^{-\gamma-1} - (2\gamma - 1)\hat{c}^{2\gamma-2} \left(\frac{\alpha}{\phi\sigma} \right)^2 \frac{1}{\hat{v}} > 0.$$

So $A(\hat{c}; \hat{v})$ is strictly convex and so can have at most two roots. *Q.E.D.*

LEMMA O.14: *Assume there are some constants λ_1, λ_2, λ_3, and a constant $\lambda_4 > 0$ such that for any feasible strategy $(\tilde{c}, a, z, \tilde{z})$ there is a nonnegative process N with*

$$dN_t \leq \left((\lambda_{1t} + \lambda_{2t}\sigma_t^N + \lambda_{3t}\tilde{\sigma}_t^N) N_t - \lambda_4 \tilde{c}_t \right) dt + \sigma_t^N N_t \, dZ_t^a + \tilde{\sigma}_t^N N_t \, d\tilde{Z}_t,$$

for some locally bounded processes σ^N and $\tilde{\sigma}^N$, which can depend on the strategy. Then for a given $T > 0$, there is a constant $\lambda_5 > 0$ such that for any feasible strategy $(\tilde{c}, a, z, \tilde{z})$,

$$\mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} \, dt \right] \leq \lambda_5 \frac{N_0^{1-\gamma}}{1-\gamma}.$$

PROOF: First, define n_t as the solution to the SDE

$$dn_t = \left((\lambda_1 + \lambda_2 \sigma_t^N + \lambda_3 \tilde{\sigma}_t^N) n_t - \tilde{c}_t \right) dt + \sigma_t^N n_t \, dZ_t^a + \tilde{\sigma}_t^N n_t \, d\tilde{Z}_t,$$

and $n_0 = \frac{N_0}{\lambda_4}$. It follows that $n_t \geq \frac{N_t}{\lambda_4} \geq 0$. Now define ζ as

$$\frac{d\zeta_t}{\zeta_t} = -\lambda_1 \, dt - \lambda_2 \, dZ_t^a - \lambda_3 \, d\tilde{Z}_t, \quad \zeta_0 = 1,$$

and

$$\tilde{n}_t = \int_0^t \zeta_s \tilde{c}_s \, ds + \zeta_t n_t.$$

We can check that $\tilde{n}_t$ is a local martingale under P^a. Since $\zeta_t > 0$ and $n_t \geq 0$, it follows that

$$\mathbb{E}^a \left[\int_0^{\tau^m \wedge T} \zeta_s \tilde{c}_s \, ds \right] \leq \mathbb{E}^a \left[\int_0^{\tau^m \wedge T} \zeta_s \tilde{c}_s \, ds + \zeta_{\tau^m \wedge T} n_{\tau^m \wedge T} \right] = n_0,$$

where $\{\tau^m\}$ reduces the stochastic integral and has $\lim_{m\to\infty}\tau^m=\infty$ a.s. Taking $m\to\infty$ and using the monotone convergence theorem, we obtain

$$\mathbb{E}^a\left[\int_0^T \zeta_s\tilde{c}_s\,ds\right]\le n_0.$$

Now we want to maximize $\mathbb{E}^a[\int_0^T e^{-\rho t}\frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}\,dt]$ subject to this budget constraint. Notice that a appears both in the budget constraint and objective function, but does not affect the law of motion of ζ under P^a, so we can ignore it since we are choosing $\tilde{c}$. The candidate solution c has

$$e^{-\rho t}c_t^{-\gamma}=\zeta_t\mu,$$

where $\mu>0$ is the Lagrange multiplier and is chosen so that the budget constraint holds with equality. For any $\tilde{c}$ that satisfies the budget constraint, we have

$$\begin{aligned}\mathbb{E}^a\left[\int_0^T e^{-\rho t}\frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}\,dt\right]&\le\mathbb{E}^a\left[\int_0^T e^{-\rho t}\left(\frac{c_t^{1-\gamma}}{1-\gamma}+c_t^{-\gamma}(\tilde{c}_t-c_t)\right)dt\right]\\&=\mathbb{E}^a\left[\int_0^T e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}\,dt\right]+\mu\mathbb{E}^a\left[\int_0^T \zeta_t(\tilde{c}_t-c_t)\,dt\right]\\&\le\mathbb{E}^a\left[\int_0^T e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}\,dt\right].\end{aligned}$$

Now since $c_t=(\zeta_t\mu)^{-\frac{1}{\gamma}}e^{-\frac{\rho}{\gamma}t}$ it follows a geometric Brownian motion so $\mathbb{E}^a[\int_0^T e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}\,dt]$ is finite. Because of homothetic preferences, we know that $\mathbb{E}^a[\int_0^T e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}\,dt]=\tilde{\lambda}_5\frac{n_0^{1-\gamma}}{1-\gamma}=\lambda_5\frac{N_0^{1-\gamma}}{1-\gamma}$ for some constant $\lambda_5>0$. *Q.E.D.*

COROLLARY: *For* $\gamma>1$, $\lim_{n\to\infty}\mathbb{E}_t^a[e^{-\rho\tau^n}\frac{N_{\tau^n}^{1-\gamma}}{1-\gamma}]=0$ *for any feasible strategy* $(\tilde{c},a,z,\tilde{z})$.

PROOF: The continuation utility at any stopping time $\tau^n<\infty$ has

$$\begin{aligned}U_{\tau^n}^{\tilde{c},a}&=\mathbb{E}_{\tau^n}^a\left[\int_{\tau^n}^{\tau^n+T}e^{-\rho(t-\tau^n)}\frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}\,dt+e^{-\rho(T-\tau^n)}U_{\tau^n+T}^{\tilde{c},a}\right]\\&\le\mathbb{E}_{\tau^n}^a\left[\int_{\tau^n}^{\tau^n+T}e^{-\rho(t-\tau^n)}\frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}\,dt\right]\le\lambda_5\frac{N_{\tau^n}^{1-\gamma}}{1-\gamma}.\end{aligned}$$

So at $t=0$ we get

$$U_0^{\tilde{c},a}=\mathbb{E}^a\left[\int_0^{\tau^n}e^{-\rho t}\frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}\,dt+e^{-\rho\tau^n}U_{\tau^n}^{\tilde{c},a}\right]\le\mathbb{E}^a\left[\int_0^{\tau^n}e^{-\rho t}\frac{\tilde{c}_t^{1-\gamma}}{1-\gamma}\,dt+e^{-\rho\tau^n}\lambda_5\frac{N_{\tau^n}^{1-\gamma}}{1-\gamma}\right].$$

Take limits $n\to\infty$ and use the monotone convergence theorem on the first term on the right-hand side to get $0\ge\lim_{n\to\infty}\mathbb{E}_t^a[e^{-\rho\tau^n}\frac{N_{\tau^n}^{1-\gamma}}{1-\gamma}]\ge 0$. *Q.E.D.*

LEMMA O.15: *Let* $\hat{c}_l \in (0, \hat{c}_h)$ *and* $\hat{v}_l \le \hat{v}_p$. *If* $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$, $\sigma^x = \frac{\alpha}{\sigma\gamma}\frac{\hat{c}_l^\gamma}{\hat{v}_l \phi}$, *and* $\tilde{\sigma}^x = \pi/\gamma$, *and* $A(\hat{c}_l, \hat{v}_l) = 0$, *where*

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2}\frac{\left(\frac{\hat{c}^\gamma \alpha}{\phi\sigma}\right)^2}{\hat{v}\gamma} + \hat{v}\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma}\right),$$

then $\hat{v}_l < \hat{c}_l^\gamma$ *and*

$$\mu^{\hat{c}} = \frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}_l^{1-\gamma}}{1-\gamma} + \frac{1}{2}(\sigma^x)^2 + \frac{1}{2}\left(\frac{\pi}{\gamma}\right)^2 > 0.$$

PROOF: Looking at (O.11), with $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$ we get for the drift

$$\mu^{\hat{c}} = \frac{r-\rho}{\gamma} + \frac{1}{2}(\sigma^x)^2 + \frac{1}{2}(\tilde{\sigma}^x)^2 - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}.$$

So $\mu^{\hat{c}} > 0$ implies

$$\frac{1}{2}(\sigma^x)^2 + \frac{1}{2}(\tilde{\sigma}^x)^2 > \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}.$$

Since we also want $A(\hat{c}; \hat{v}) = 0$, we get

$$\begin{aligned} 0 &= \hat{c} - r\hat{v} + \hat{v}\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}(\sigma^x)^2 - \frac{\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right) \\ &< \hat{c} - \hat{v}\hat{c}^{1-\gamma} \equiv M. \end{aligned}$$

Notice that if $\hat{v} = \hat{c}^\gamma$ we have $M = 0$. If $\hat{v} > \hat{c}^\gamma$, we have $M < 0$ and if $\hat{v} < \hat{c}^\gamma$ we have $M > 0$. So for $A(\hat{c}; \hat{v}) = 0$ and $\mu^{\hat{c}} > 0$ we need $\hat{v} < \hat{c}^\gamma$. In fact, if $\hat{v} = \hat{c}^\gamma$ and in addition

$$\frac{1}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2 + \frac{1}{2}\left(\frac{\pi}{\gamma}\right)^2 = \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\rho - r}{\gamma}, \tag{O.37}$$

then we have $A = 0$ and $\mu^{\hat{c}} = 0$. In this case, because we have $\mu^{\hat{c}} = 0$ we therefore have the value of a stationary contract, that is, $\hat{v} = \hat{v}_s(\hat{c})$ given by (O.32). This point corresponds to the myopic stationary contract with $(\hat{c}_p, \hat{v}_p)$. We know from Lemma O.16 that $\hat{c}_p \in [\hat{c}_a, \hat{c}_h]$. By assumption, $\hat{v}_l \le \hat{v}_p$.

First, we will show that $\mu^{\hat{c}} \ge 0$, and then make the inequality strict. Toward contradiction, suppose $\mu^{\hat{c}} < 0$ at $\hat{c}_l$. Then it must be the case that $\hat{v}_l > \hat{c}_l^\gamma$ because we have $A(\hat{c}_l, \hat{v}_l) = 0$. We will show that $A(\hat{c}_l, \hat{v}_l) > 0$ and get a contradiction. First, take the derivative of A:

$$A'_{\hat{c}}(\hat{c}_l, \hat{v}_l) = 1 - \hat{v}_l\left(\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1}\left(\frac{\alpha}{\phi\sigma}\right)^2 \frac{1}{\hat{v}_l^2}\right) < 0,$$

where the inequality holds for all $\hat{c} < \hat{v}_l^{\frac{1}{\gamma}}$. So $A(\hat{c}_l, \hat{v}_l) > A(\hat{v}_l^{\frac{1}{\gamma}}, \hat{v}_l)$. Letting $\hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}}$, we get

$$\begin{aligned} A(\hat{c}_l, \hat{v}_l) &> \hat{c}_m - r\hat{v}_l + \hat{v}_l\left(\frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \frac{1}{2}\left(\frac{\alpha}{\phi\sigma}\right)^2\frac{1}{\gamma} - \frac{\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right) \\ &= \hat{c}_m - r\hat{v}_l + \hat{v}_l\left(\frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \gamma\frac{\rho - \hat{c}_p^{1-\gamma}}{1-\gamma} - (\rho - r)\right) \\ \implies \quad A(\hat{c}_l, \hat{v}_l) &> \hat{c}_m + \hat{v}_l\frac{\gamma\hat{c}_p^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} = \hat{c}_m^{\gamma}\gamma\frac{\hat{c}_p^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} \geq 0, \end{aligned}$$

where the last equality uses $\hat{v}_l = \hat{c}_m^{\gamma}$ and the last inequality uses $\hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}} \leq \hat{v}_p^{\frac{1}{\gamma}} = \hat{c}_p$. This is a contradiction and, therefore, it must be the case that $\mu^{\hat{c}} \geq 0$ at $\hat{c}_l$.

It is clear from the previous argument that $\mu^{\hat{c}}(\hat{c}_l) = 0$ only if $(\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p)$. We will show this cannot be the case because $\alpha > 0$. First, note that $(\hat{c}_p, \hat{v}_p)$ is a tangency point where $\hat{v}_s(\hat{c})$ touches the locus $\hat{v}_m(\hat{c})$ defined by $A(\hat{c}; \hat{v}_m(\hat{c})) = 0$. If $(\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p)$, then this must be the minimum point for $\hat{v}_s(\hat{c})$, so the derivative of both $\hat{v}_s(\hat{c})$ and $\hat{v}_m(\hat{c})$ must be zero. This means that $A'_{\hat{c}}(\hat{c}_l, \hat{v}_l) = 0$. However,

$$1 - \hat{v}_l\left(\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1}\left(\frac{\alpha}{\phi\sigma}\right)^2\frac{1}{\hat{v}_l^2}\right) < 0,$$

where the inequality follows from $\hat{v}_l = \hat{v}_p = \hat{c}_l^{\gamma}$ (note that $\hat{c}_l > 0$ because as Lemma O.13 shows $A(\hat{c}, \hat{v}_l)$ is strictly positive for $\hat{c}$ near 0). This cannot be a minimum of $\hat{v}_s(\hat{c})$. Therefore, $(\hat{c}_l, \hat{v}_l) \neq (\hat{c}_p, \hat{v}_p)$ and $\mu^{\hat{c}}(\hat{c}_l) > 0$. This completes the proof. *Q.E.D.*

LEMMA O.16: *Let*

$$\begin{aligned} \hat{c}_p &\equiv \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2 - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}}, \\ \hat{v}_p &\equiv \hat{c}_p^{\gamma} \end{aligned}$$

be the $\hat{c}$ and $\hat{v}$ corresponding to the myopic stationary contract. We have the following properties:

1. $\hat{c}_a < \hat{c}_p < \hat{c}_r \leq \hat{c}_h$, *for any valid hidden investment setting*
2. $\hat{c}^{\gamma}$ *intersects* $\hat{v}_s(\hat{c})$ *only at* $\hat{c}_p$ *and* $\hat{c}_u = (\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2)^{\frac{1}{1-\gamma}}$ *in* $[0, \hat{c}_u]$. *Furthermore,* $\hat{c}^{\gamma} \geq \hat{v}_s(\hat{c})$ *for all* $\hat{c} \in [\hat{c}_p, \hat{c}_u]$, *and* $\hat{c}^{\gamma} \leq \hat{v}_s(\hat{c})$ *for all* $\hat{c} \in [\hat{c}_a, \hat{c}_p]$, *with strict inequality in the interior of each region.*
3. $A(\hat{c}, \hat{c}^{\gamma}) = 0$ *only at* $\hat{c} = 0$ *and* $\hat{c}_p$. *Furthermore,* $A(\hat{c}, \hat{c}^{\gamma}) \leq 0$ *for all* $\hat{c} \in [\hat{c}_p, \hat{c}_h]$ *and* $A(\hat{c}, \hat{c}^{\gamma}) \geq 0$ *for all* $\hat{c} \in [0, \hat{c}_p]$, *and* $\partial_1 A(\hat{c}, \hat{c}^{\gamma}) < 0$ *for all* $\hat{c} \in (0, \hat{c}_h]$.

PROOF: First, let us show that $\hat{c}_p \in (\hat{c}_a, \hat{c}_h)$. Clearly, $\hat{c}_p < \hat{c}_h$ for any type of valid hidden investment, because $\phi < 1$. Now write $\hat{c}_p$

$$\begin{aligned}\hat{c}_p &= \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2 - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}} \\ &> \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}}\left(1 - \frac{1-\gamma}{1+\gamma}\right)^{\frac{1}{1-\gamma}},\end{aligned}$$

where the inequality comes from $\alpha < \bar{\alpha} = \frac{\phi\sigma\gamma\sqrt{2}}{\sqrt{1+\gamma}}\sqrt{\frac{\rho-r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2}$. Notice $1 - \frac{1-\gamma}{1+\gamma} = \frac{2\gamma}{1+\gamma}$ and use the definition of $\hat{c}_a$,

$$\hat{c}_a = \left(\frac{2\gamma}{1+\gamma}\right)^{\frac{1}{1-\gamma}}\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}},$$

to conclude that $\hat{c}_a < \hat{c}_p$. The cost of this contract is $\hat{v}_p = \hat{c}_p^{\gamma}$.

Now go to 2). We are looking for roots of $\hat{v}_s(\hat{c}) = \hat{c}^{\gamma}$:

$$\begin{aligned}&\hat{c} - \frac{\alpha}{\phi\sigma}\hat{c}^{\gamma}\sqrt{2}\sqrt{\frac{\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right) - \hat{c}^{1-\gamma}}{1-\gamma}} \\ &\quad = \hat{c}^{\gamma}\left(2r - \rho - \frac{1+\gamma}{1-\gamma}\rho + \gamma\left(\frac{\pi}{\gamma}\right)^2 + \frac{\hat{c}^{1-\gamma}}{1-\gamma}(1+\gamma)\right).\end{aligned}$$

Divide throughout by $\hat{c}^{\gamma} > 0$ and reorganize the right-hand side

$$\begin{aligned}&\frac{\hat{c}^{1-\gamma}}{1-\gamma}(1-\gamma) - \frac{\alpha}{\phi\sigma}\sqrt{2}\sqrt{\frac{\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right) - \hat{c}^{1-\gamma}}{1-\gamma}} \\ &\quad = -2\gamma\frac{\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)}{1-\gamma} + \frac{\hat{c}^{1-\gamma}}{1-\gamma}(1+\gamma), \\ &-\frac{\alpha}{\phi\sigma}\sqrt{2}\sqrt{\frac{\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right) - \hat{c}^{1-\gamma}}{1-\gamma}} \\ &\quad = -2\gamma\frac{\left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)}{1-\gamma} + \frac{\hat{c}^{1-\gamma}}{1-\gamma}2\gamma,\end{aligned}$$

$$\frac{\alpha}{\phi\sigma}\sqrt{2}\sqrt{\frac{\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)-\hat{c}^{1-\gamma}}{1-\gamma}}$$
$$=2\gamma\frac{\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)-\hat{c}^{1-\gamma}}{1-\gamma}.$$

If $\hat{c}=(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}(\frac{\pi}{\gamma})^2)^{\frac{1}{1-\gamma}}$, we have a root. If not, then we can write

$$\frac{\alpha}{\phi\sigma\gamma}=\sqrt{2}\sqrt{\frac{\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)-\hat{c}^{1-\gamma}}{1-\gamma}},$$
$$\hat{c}=\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}}$$
$$=\hat{c}_p<\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)^{\frac{1}{1-\gamma}}.$$

We know that at $\hat{c}=0$, $\hat{c}^{\gamma}=0$, while $\hat{v}_s(\hat{c})$ is always positive above $\hat{c}_a$ and diverges to infinity as $\hat{c}\searrow\hat{c}_a$. So we know that $\hat{c}_p$ is the first time they intersect and, therefore, $\hat{c}^{\gamma}$ intersects $\hat{v}_s(\hat{c})$ from below. Since they will not intersect again until $\hat{c}_u$, we get the other inequality.

Back to (1), consider the locus $\hat{v}_m(\hat{c})$ defined by $A(\hat{c},\hat{v}_m(\hat{c}))=0$. Since $A(\hat{c},\hat{v})$ minimizes over σ^x, it is always below $\hat{v}_s(\hat{c})$. At $(\hat{c}_p,\hat{v}_p)$ we have $\hat{v}_m(\hat{c})=\hat{v}_s(\hat{c})$ by part (3) below, which means this is a tangency point of $\hat{v}_m$ and $\hat{v}_s$. We can now show that $A'_{\hat{c}}(\hat{c}_p,\hat{v}_p)<0$ and $A'_{\hat{v}}(\hat{c}_p,\hat{v}_p)<0$, so that $\hat{v}_m(\hat{c}_p)=\hat{v}_s(\hat{c}_p)<0$ which means that the $\mathcal{C}_p$ is not the optimal stationary contract, since $\hat{c}_p<\hat{c}_h$. Write

$$A'_{\hat{c}}(\hat{c}_p,\hat{v}_p)=1-\hat{v}_p\left(\hat{c}_p^{-\gamma}+\hat{c}_p^{2\gamma-1}\left(\frac{\alpha}{\phi\sigma}\right)^2\frac{1}{\hat{v}_p^2}\right)=-\hat{c}_p^{\gamma-1}\left(\frac{\alpha}{\phi\sigma}\right)<0,$$

$$A'_{\hat{v}}(\hat{c}_p,\hat{v}_p)=\frac{1}{1-\gamma}\left(\gamma\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)-\hat{c}_p^{1-\gamma}\right)+\frac{1}{2}\frac{\left(\frac{\hat{c}_p^{\gamma}\alpha}{\phi\sigma}\right)^2}{\hat{v}_p^2\gamma}$$
$$=\frac{1}{1-\gamma}\left(\gamma\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)\right.$$
$$\left.-\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)\right)$$
$$+\frac{\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2$$
$$=\frac{1}{1-\gamma}\left((\gamma-1)\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)\right)+\frac{1+\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2<0,$$

where the last inequalities follows from the bound on

$$\alpha < \bar{\alpha} \equiv \frac{\phi\sigma\gamma\sqrt{2}}{\sqrt{1+\gamma}}\sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2}.$$

To find the best stationary contract, use the HJB

$$r\hat{v}_r = \min_{\hat{c}} \hat{c} - \sigma_s^x(\hat{c})\hat{c}^\gamma \frac{\alpha}{\phi\sigma} + \hat{v}_r\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2}\left(\sigma_s^x(\hat{c})\right)^2 + \frac{\gamma}{2}(\pi/\gamma)^2\right),$$

with FOC for $\hat{c}$:

$$1 - \gamma\hat{c}^{\gamma-1}\frac{\alpha}{\phi\sigma}\sigma_s^x(\hat{c}) - \hat{v}_r\hat{c}^{-\gamma} + \left(\hat{v}_r\gamma\sigma_s^x(\hat{c}) - \hat{c}^\gamma\frac{\alpha}{\phi\sigma}\right)\partial_{\hat{c}}\sigma_s^x(\hat{c}) = 0.$$

We already know that for $\hat{c} \leq \hat{c}_p$ we have $\hat{v}_s(\hat{c}) \geq \hat{c}^\gamma$ and $\sigma_s^x(\hat{c}) \geq \frac{\alpha}{\gamma\phi\sigma}$. We can them show that for $\hat{c} \leq \hat{c}_p$ the left-hand side of the FOC is strictly negative:

$$lhs = 1 - \gamma\hat{c}^{\gamma-1}\frac{\alpha}{\phi\sigma}\sigma_s^x(\hat{c}) - \hat{v}_s(\hat{c})\hat{c}^{-\gamma} + \left(\hat{v}_s(\hat{c})\gamma\sigma_s^x(\hat{c}) - \hat{c}^\gamma\frac{\alpha}{\phi\sigma}\right)\partial_{\hat{c}}\sigma_s^x(\hat{c}).$$

Use $\hat{v}_s(\hat{c}) \geq \hat{c}^\gamma$ and $\partial_{\hat{c}}\sigma_s^x(\hat{c}) < 0$ to obtain

$$lhs \leq -\gamma\hat{c}^{\gamma-1}\frac{\alpha}{\phi\sigma}\sigma_s^x(\hat{c}) + \left(\hat{v}_s(\hat{c})\gamma\sigma_s^x(\hat{c}) - \hat{c}^\gamma\frac{\alpha}{\phi\sigma}\right)\partial_{\hat{c}}\sigma_s^x(\hat{c})$$

and

$$lhs \leq -\gamma\hat{c}^{\gamma-1}\frac{\alpha}{\phi\sigma}\sigma_s^x(\hat{c}) + \hat{c}^\gamma\left(\gamma\sigma_s^x(\hat{c}) - \frac{\alpha}{\phi\sigma}\right)\partial_{\hat{c}}\sigma_s^x(\hat{c}).$$

Finally, $\sigma_s^x(\hat{c}) \geq \frac{\alpha}{\gamma\phi\sigma}$ yields $lhs < 0$. This means the best stationary contract must have $\hat{c}_r > \hat{c}_p$. We know $\hat{c}_r \leq \hat{c}_h$ from the definition of $\hat{c}_r$.

For (3), we are looking for roots of

$$\hat{c} - r\hat{c}^\gamma - \frac{1}{2}\frac{\left(\frac{\alpha\hat{c}^\gamma}{\phi\sigma}\right)^2}{\hat{c}^\gamma\gamma} + \hat{c}^\gamma\left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma}\right) = 0.$$

This works for $\hat{c} = 0$. Otherwise, divide by $\hat{c}^\gamma$,

$$\frac{\hat{c}^{1-\gamma}}{1-\gamma}(1-\gamma) - r - \frac{1}{2}\frac{\left(\frac{\alpha}{\phi\sigma}\right)^2}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2}\frac{\pi^2}{\gamma} = 0,$$

$$\frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2 - \frac{\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2 = \frac{\hat{c}^{1-\gamma}}{1-\gamma}\gamma,$$

$$\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2}\left(\frac{\alpha}{\phi\sigma\gamma}\right)^2 - \frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2 = \hat{c}^{1-\gamma},$$

$$\hat{c} = \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1-\gamma}{2} \left(\frac{\alpha}{\phi \sigma \gamma} \right)^2 - \frac{1-\gamma}{2} \left(\frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} = \hat{c}_p.$$

So we have only $\hat{c}_p$ and $\hat{c} = 0$ as roots. This argument also shows that $A(\hat{c}, \hat{c}^\gamma) \leq 0$ for $\hat{c} \in [\hat{c}_p, \hat{c}_h]$, and $A(\hat{c}, \hat{c}^\gamma) \geq 0$ for $\hat{c} \in [0, \hat{c}_p]$. Also, evaluating the derivative $\partial_1 A(\hat{c}, \hat{c}^\gamma)$,

$$\partial_1 A(\hat{c}, \hat{c}^\gamma) = 1 - \hat{c}^\gamma \hat{c}^{-\gamma} - \hat{c}^{2\gamma - 1} \left(\frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{c}^\gamma},$$

$$\partial_1 A(\hat{c}, \hat{c}^\gamma) = 1 - 1 - \hat{c}^{\gamma - 1} \left(\frac{\alpha}{\phi \sigma} \right)^2 = -\hat{c}^{\gamma - 1} \left(\frac{\alpha}{\phi \sigma} \right)^2 < 0$$

for all $\hat{c} \in (0, \hat{c}_h]$. *Q.E.D.*

LEMMA O.17: *Suppose $\mu^{\hat{c}}$ and $\sigma^{\hat{c}}$ are derived from first-order conditions from a solution to the HJB equation* (O.20) *with the properties in Theorem* O.2. *Without hidden investment*, $H = \{0\}$, *the drift and volatility of $\hat{c}$ near $\hat{c}_h$ are approximately*,

$$\mu^{\hat{c}} \hat{c} \approx \left(4\gamma - 6(1+\gamma)^2\right) \hat{c}_h^{-\gamma} \epsilon,$$

$$\sigma^{\hat{c}} \hat{c} \approx -\sqrt{2} 2(1+\gamma) \hat{c}_h^{-\gamma/2} \epsilon^{3/2},$$

where $\epsilon = \hat{c}_h - \hat{c}$. With hidden investment, $H = \mathbb{R}^+$, *we have*

$$\mu^{\hat{c}} \hat{c} \approx (\eta - 2) \frac{1}{2} \left(\frac{\alpha}{\sigma \gamma} \right)^2 \left(\frac{\gamma}{1 - \eta} \right)^2 \epsilon < 0,$$

$$\sigma^{\hat{c}} \hat{c} \approx -\left(\frac{\alpha}{\sigma \gamma} \right) \frac{\gamma}{1 - \eta} \epsilon,$$

with $\eta \in (0, 1)$.

PROOF: WITHOUT HIDDEN INVESTMENT. First, we derive the drift of $\hat{v}'$ using the HJB equation (21). Differentiating with respect to $\hat{c}$ and using the envelope theorem, we obtain

$$\begin{aligned} r\hat{v}' &= 1 - \gamma \sigma^x \hat{c}^{\gamma - 1} \frac{\alpha}{\phi \sigma} + \hat{v}' \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} \left(\sigma^x\right)^2 - \frac{\gamma}{2} \left(\frac{\pi}{\gamma} \right)^2 \right) - \hat{v} \hat{c}^{-\gamma} \\ &\quad + \hat{v}'' \hat{c} \left(\frac{\hat{c}^{1-\gamma} - \hat{c}_h^{1-\gamma}}{1-\gamma} + \frac{\left(\sigma^x\right)^2}{2} + (1+\gamma) \sigma^x \sigma^{\hat{c}} + \frac{1+\gamma}{2} \left(\sigma^{\hat{c}}\right)^2 \right) + \frac{\hat{v}'''}{2} \hat{c}^2 \left(\sigma^{\hat{c}}\right)^2 \\ &\quad + \hat{v}' \hat{c}^{1-\gamma} + \hat{v}' \left(\frac{\hat{c}^{1-\gamma} - \hat{c}_h^{1-\gamma}}{1-\gamma} + \frac{\left(\sigma^x\right)^2}{2} + (1+\gamma) \sigma^x \sigma^{\hat{c}} + \frac{1+\gamma}{2} \left(\sigma^{\hat{c}}\right)^2 \right) + \hat{v}'' \hat{c} \left(\sigma^{\hat{c}}\right)^2. \end{aligned}$$

The middle line has the drift of $\hat{v}'$ plus an extra term $\hat{v}'' \hat{c} \sigma^x \sigma^{\hat{c}}$, which we can combine with the term containing $\hat{v}''$ in the third line. We also know that $\hat{v}'' \sigma^{\hat{c}} \hat{c} = -\hat{v}'(1+\gamma)(\sigma^{\hat{c}} + \sigma^x)$

from the FOC for $\sigma^{\hat{c}}$. Using this, we find the drift of $\hat{v}'$ to be

$$
\begin{aligned}
&+\hat{v}''\hat{c}\left(\frac{\hat{c}^{1-\gamma}-\hat{c}_h^{1-\gamma}}{1-\gamma}+\frac{(\sigma^x)^2}{2}+\gamma\sigma^x\sigma^{\hat{c}}+\frac{1+\gamma}{2}(\sigma^{\hat{c}})^2\right)+\frac{\hat{v}'''}{2}\hat{c}^2(\sigma^{\hat{c}})^2 \\
&\quad=\hat{v}'\left(\frac{1+\gamma}{2}(\sigma^x+\sigma^{\hat{c}})^2+\hat{c}_h^{1-\gamma}-\hat{c}^{1-\gamma}\right)+\gamma\sigma^x\hat{c}^{\gamma-1}\frac{\alpha}{\phi\sigma}+\hat{v}\hat{c}^{-\gamma}-1.
\end{aligned}
$$

Now we approximate the cost function near $\hat{c}_h$. Conjecture, and later verify, that $\hat{v}(\hat{c})=\hat{c}_h^{\gamma}-K\sqrt{\epsilon}$. Then

$$
\hat{v}'=\frac{K}{2}\epsilon^{-1/2},\qquad \hat{v}''=\frac{K}{4}\epsilon^{-3/2},\qquad \hat{v}'''=\frac{3K}{8}\epsilon^{-5/2},
$$

plus smaller order terms.

Now conjecture that $\sigma^{\hat{c}}$ is of smaller order than σ^x (also verified later) and use the FOC for σ^x to obtain

$$
\hat{c}^{\gamma}\frac{\alpha}{\phi\sigma}=\frac{K}{2}\epsilon^{-1/2}\hat{c}\sigma^x \quad\Longrightarrow\quad \sigma^x=\frac{2}{K}\hat{c}^{\gamma-1}\frac{\alpha}{\phi\sigma}\sqrt{\epsilon},
$$

plus smaller order terms. Now plug into the FOC for $\sigma^{\hat{c}}$:

$$
\frac{K}{2}\epsilon^{-1/2}(1+\gamma)(\sigma^x+\sigma^{\hat{c}})+\frac{K}{4}\epsilon^{-3/2}\hat{c}\sigma^{\hat{c}}=0 \quad\Longrightarrow\quad \hat{c}\sigma^{\hat{c}}=-2(1+\gamma)\sigma^x\epsilon.
$$

This verifies that indeed $\sigma^{\hat{c}}$ is of smaller order than σ^x.

Now we plug everything into the HJB equation and collect terms of order $\sqrt{\epsilon}$ (the constant order terms match because the cost function we specified works at $\hat{c}_h$). The only terms of order $\sqrt{\epsilon}$ are

$$
\begin{aligned}
&-\sigma^x\hat{c}^{\gamma}\frac{\alpha}{\phi\sigma}+\left(\hat{c}+\hat{v}\frac{\gamma\hat{c}_h^{1-\gamma}-\hat{c}^{1-\gamma}}{1-\gamma}\right)+\hat{v}'\hat{c}\left(\frac{(\sigma^x)^2}{2}-\frac{\hat{c}_h^{1-\gamma}-\hat{c}^{1-\gamma}}{1-\gamma}\right)=0, \\
&-2\hat{c}_h^{2\gamma-1}\left(\frac{\alpha}{\phi\sigma}\right)^2-K^2\frac{\gamma\hat{c}_h^{1-\gamma}-\hat{c}_h^{1-\gamma}}{1-\gamma}+\frac{K}{2}\hat{c}_h\left(\frac{4}{2}\frac{1}{K^2}\hat{c}_h^{2\gamma-2}-\hat{c}_h^{-\gamma}\right)=0.
\end{aligned}
$$

We can solve for

$$
K=\sqrt{2}\hat{c}_h^{1.5\gamma-1}\frac{\alpha}{\phi\sigma}.
$$

Now we plug into our expression for σ^x and $\sigma^{\hat{c}}$

$$
\begin{aligned}
\sigma^x&=\sqrt{2}\hat{c}_h^{-\gamma/2}\sqrt{\epsilon}, \\
\sigma^{\hat{c}}\hat{c}&=-\sqrt{2}2(1+\gamma)\hat{c}_h^{-\gamma/2}\epsilon^{3/2},
\end{aligned}
$$

as desired.

For the drift, evaluate the drift of $\hat{v}'$ using the formula above

$$
\begin{aligned}
&\frac{K}{2}\epsilon^{-1/2}\left(\frac{1+\gamma}{2}2\hat{c}_h^{-\gamma}\epsilon+(1-\gamma)\hat{c}_h^{-\gamma}\epsilon\right)+\gamma\left(\sqrt{2}\hat{c}_h^{-\gamma/2}\sqrt{\epsilon}\right)\hat{c}^{\gamma-1}\frac{\alpha}{\phi\sigma}+\left(\hat{c}_h^{\gamma}-K\sqrt{\epsilon}\right)\hat{c}^{-\gamma}-1 \\
&\quad=\gamma K\hat{c}_h^{-\gamma}\sqrt{\epsilon}.
\end{aligned}
$$

But we can also use Ito's lemma to obtain the drift of $\hat{v}'$

$$\hat{v}''\hat{c}\mu^{\hat{c}} + \frac{1}{2}\hat{v}'''\big(\hat{c}\sigma^{\hat{c}}\big)^2 = \frac{K}{4}\epsilon^{-3/2}\hat{c}\mu^{\hat{c}} + \frac{1}{2}\frac{3K}{8}\epsilon^{-5/2}8(1+\gamma)^2\hat{c}_h^{-\gamma}\epsilon^3 = \gamma K\hat{c}_h^{-\gamma}\sqrt{\epsilon}.$$

Solve for $\hat{c}\mu^{\hat{c}}$

$$\hat{c}\mu^{\hat{c}} = \big(4\gamma - 6(1+\gamma)^2\big)\hat{c}_h^{-\gamma}\epsilon^2,$$

which completes the proof.

WITH HIDDEN INVESTMENT. The IC constraints for hidden investment will be binding near $\hat{c}_h$, so we have

$$\sigma^x = \frac{\hat{c}^\gamma\dfrac{\alpha}{\phi\sigma} + \hat{v}''\hat{c}^2\dfrac{\alpha}{\sigma\gamma}}{\gamma\big(\hat{v} - \hat{v}'\hat{c}\big) + \hat{v}''\hat{c}^2},$$

$$\sigma^{\hat{c}} = \frac{\alpha}{\gamma\sigma} - \sigma^x.$$

In this case, we use the approximation

$$\hat{v} = \hat{v}_h - K\epsilon^\eta,$$

$$\hat{v}' = K\eta\epsilon^{\eta-1},$$

$$\hat{v}'' = -K\eta(\eta-1)\epsilon^{\eta-2}.$$

Divide the FOC for σ^x by $\hat{v}''\hat{c}$ on both sides ($\hat{v}'' \neq 0$, or we would have $\sigma^x > \alpha/(\gamma\sigma)$ and the IC wouldn't be binding):

$$\sigma^x = \frac{\dfrac{\alpha}{\sigma\gamma} + \dfrac{\hat{c}^\gamma\dfrac{\alpha}{\phi\sigma}}{K\eta(\eta-1)}\epsilon^{2-\eta}}{1 + \dfrac{\gamma\big(\hat{v}_s - K\epsilon^\eta - K\eta\epsilon^{\eta-1}\hat{c}\big)}{K\eta(\eta-1)\hat{c}^2}\epsilon^{2-\eta}}.$$

The largest terms are of order ϵ because $\eta \in (0,1)$, so we get

$$\sigma^x \approx \frac{\alpha}{\sigma\gamma}(1 + A\epsilon),$$

where $A = \gamma\hat{c}_h^{-1}\frac{1}{1-\eta} > 0$ and, therefore,

$$\sigma^{\hat{c}} \approx -\frac{\alpha}{\sigma\gamma}A\epsilon.$$

We need to make sure the HJB holds up to terms of order ϵ^η. Plug into the HJB to obtain

$$0 = (\hat{c}_h - \epsilon) - \left(\frac{\alpha}{\sigma\gamma}\right)(1 + A\epsilon)\big(\hat{c}_h^\gamma - \gamma\hat{c}_h^{\gamma-1}\epsilon\big)\frac{\alpha}{\phi\sigma}$$

$$+ \big(\hat{v}_h - K\epsilon^\eta\big)\left(\frac{\gamma\left(\dfrac{\rho - r(1-\gamma)}{\gamma} - \dfrac{1-\gamma}{2}\left(\dfrac{\pi}{\gamma}\right)^2\right) - \hat{c}_h^{1-\gamma}}{1-\gamma}\right.$$

$$+\hat{c}_h^{-\gamma}\epsilon+\frac{\gamma}{2}\left(\frac{\alpha}{\sigma\gamma}\right)^2(1+A\epsilon)^2\Bigg)$$

$$+K\eta\epsilon^{\eta-1}(\hat{c}_h-\epsilon)\left(\left(\frac{\alpha}{\gamma\sigma}\right)^2\left(\frac{(1+A\epsilon)^2}{2}-(1+\gamma)(1+A\epsilon)A\epsilon+\frac{1+\gamma}{2}A^2\epsilon^2\right)\right.$$

$$\left.-\frac{\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)-\hat{c}_h^{1-\gamma}}{1-\gamma}-\hat{c}_h^{-\gamma}\epsilon\right)$$

$$-K\eta(\eta-1)\epsilon^{\eta-2}\left(\hat{c}_h^2-2\hat{c}_h\epsilon\right)\left(\frac{\alpha}{\gamma\sigma}\right)^2A^2\epsilon^2.$$

The constant terms match. Then there are terms of order $\epsilon^{\eta-1}$:

$$K\eta\hat{c}_h\left(\left(\frac{\alpha}{\gamma\sigma}\right)^2\frac{1}{2}-\frac{\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)-\hat{c}_h^{1-\gamma}}{1-\gamma}\right)$$

$$=K\eta\hat{c}_h\left(\left(\frac{\alpha}{\gamma\sigma}\right)^2\frac{1}{2}-\frac{\frac{1-\gamma}{2}\left(\frac{\alpha}{\gamma\sigma}\right)^2}{1-\gamma}\right)=0.$$

Then there are terms of order ϵ^{η}:

$$-K\left(\frac{\gamma\left(\frac{\rho-r(1-\gamma)}{\gamma}-\frac{1-\gamma}{2}\left(\frac{\pi}{\gamma}\right)^2\right)-\hat{c}_h^{1-\gamma}}{1-\gamma}+\frac{\gamma}{2}\left(\frac{\alpha}{\sigma\gamma}\right)^2\right)$$

$$+K\eta\hat{c}_h\left(\left(\frac{\alpha}{\gamma\sigma}\right)^2\big(A-(1+\gamma)A\big)-\hat{c}_h^{-\gamma}\right)$$

$$-\frac{1}{2}K\eta(\eta-1)\hat{c}_h^2\left(\frac{\alpha}{\gamma\sigma}\right)^2A^2.$$

We want this to be zero. K factors out, and there is a unique $\eta\in(0,1)$ that makes this expression zero. After some algebra, we obtain

$$\hat{c}_h^{1-\gamma}(1-\eta)^2+\eta\left(1-\frac{\gamma}{2}\right)\gamma\left(\frac{\alpha}{\sigma\gamma}\right)^2-\gamma\left(\frac{\alpha}{\sigma\gamma}\right)^2=0. \tag{O.38}$$

The bound $\alpha<\bar{\alpha}$ implies $\hat{c}_h^{1-\gamma}>\gamma(\frac{\alpha}{\gamma\sigma})^2>0$, so at $\eta=0$ the rhs is strictly positive. At $\eta=1$, we have $\gamma(\frac{\alpha}{\sigma\gamma})^2-(1-\frac{\gamma}{2})\gamma(\frac{\alpha}{\sigma\gamma})^2=\frac{\gamma^2}{2}(\frac{\alpha}{\sigma\gamma})^2>0$, so the rhs is negative. And because the vertex of the quadratic term is $\eta=1$ there is a unique η that satisfies the expression. So we have

$$\sigma^x=\left(\frac{\alpha}{\sigma\gamma}\right)\left(1+\gamma\hat{c}_h^{-1}\frac{1}{1-\eta}\epsilon\right),$$

$$\sigma^{\hat{c}}\hat{c} \approx -\left(\frac{\alpha}{\sigma\gamma}\right)\frac{\gamma}{1-\eta}\epsilon.$$

Now let us find the drift of $\hat{c}$. Using (17) and plugging in the expression for σ^x and $\sigma^{\hat{c}}$, the constant terms cancel, and we get terms of order ϵ (plus smaller terms)

$$\mu^{\hat{c}}\hat{c} = \left(-\hat{c}_h^{1-\gamma} + \left(\frac{\alpha}{\sigma\gamma}\right)^2 \frac{\gamma}{1-\eta}(1-\gamma)\right)\epsilon.$$

Now use (O.38) to replace $\hat{c}_h^{1-\gamma}$ and obtain

$$\mu^{\hat{c}}\hat{c} = \left(\frac{\alpha}{\sigma\gamma}\right)^2\left(\frac{\gamma}{1-\eta}(1-\gamma) + \eta\frac{2-\gamma}{2}\frac{\gamma}{(1-\eta)^2} - \frac{\gamma}{(1-\eta)^2}\right).$$

After some algebra, we get

$$\mu^{\hat{c}}\hat{c} \approx \frac{\eta-2}{2}\left(\frac{\alpha}{\sigma\gamma}\right)^2\left(\frac{\gamma}{1-\eta}\right)^2\epsilon < 0,$$

as desired. *Q.E.D.*

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