Econometrica Supplementary Material

SUPPLEMENT TO "ON THE INFORMATIVENESS OF DESCRIPTIVE STATISTICS FOR STRUCTURAL ESTIMATES"

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ISAIAH ANDREWS
Department of Economics, Harvard University and NBER

MATTHEW GENTZKOW
Department of Economics, Stanford University and NBER

JESSE M. SHAPIRO
Department of Economics, Brown University and NBER

APPENDIX A: SENSITIVITY AND INFORMATIVENESS

PROPOSITION 2 considers the effect of limiting attention to forms of misspecification that do not affect $\hat{\gamma}$. In some cases, however, researchers may be interested in forms of misspecification with a non-zero, but known, effect on $\hat{\gamma}$. In such cases, our assumptions again imply a relationship between the biases in $\hat{c}$ and $\hat{\gamma}$.

This relationship depends on the sensitivity of $\hat{c}$ to $\hat{\gamma}$. This is the natural extension of the sensitivity measure proposed in Andrews, Gentzkow, and Shapiro (2017) to the current setting.

DEFINITION: The ***sensitivity*** of $\hat{c}$ with respect to $\hat{\gamma}$ is

$$\Lambda = \Sigma_{c\gamma}\Sigma_{\gamma\gamma}^{-1}.$$

To build intuition, note that sensitivity characterizes the relationship between $\hat{c}$ and $\hat{\gamma}$ in the asymptotic distribution under the base model. If we assume, as in Section 3, that $\hat{c}$ and $\hat{\gamma}$ are normally distributed in finite samples, then Λ is simply the vector of coefficients from the population regression of $\hat{c}$ on $\hat{\gamma}$. In this case, element Λ_j of Λ is the effect of changing the realization of a particular $\hat{\gamma}_j$ on the expected value of $\hat{c}$, holding the other elements of $\hat{\gamma}$ constant.

Andrews, Gentzkow, and Shapiro (2017) showed that for $\hat{c} = c(\hat{\eta})$, $\hat{\eta}$ a minimum distance estimator based on moments $\hat{g}(\eta)$, and $\hat{\gamma} = \hat{g}(\eta_0)$ the estimation moments evaluated at the true parameter value, under regularity conditions sensitivity translates the effect of misspecification on $\hat{\gamma}$ to the effect on $\hat{c}$, in the sense that

$$\bar{c}\big(S(h,z)\big) - \bar{c}\big(S(h,0)\big) = \Lambda\big(\bar{\gamma}\big(S(h,z)\big) - \bar{\gamma}\big(S(h,0)\big)\big).$$

Our next proposition extends this result.

PROPOSITION 4: *Suppose that Assumptions* 1–4 *hold*, *and let*

$$\mathcal{S}^{\mathrm{RN}}(c^\star, \bar{\gamma}) = \bigcup_{S\in\mathcal{S}^0(c^\star)} \{\tilde{S} \in \mathcal{N}(S) : \bar{\gamma}(\tilde{S}) - \bar{\gamma}(S) = \bar{\gamma}\}.$$

Isaiah Andrews: iandrews@fas.harvard.edu
Matthew Gentzkow: gentzkow@stanford.edu
Jesse M. Shapiro: jesse_shapiro_1@brown.edu

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Provided $\mu(\bar{\gamma})^2 = \mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma} \geq 0$, *the set of possible biases under* $S \in \mathcal{S}^{\mathrm{RN}}(\cdot, \bar{\gamma})$ *is*

$$\left\{\bar{c}(S) - c^\star : S \in \mathcal{S}^{\mathrm{RN}}\left(c^\star, \bar{\gamma}\right)\right\} = \left[\Lambda\bar{\gamma} - \mu(\bar{\gamma})\sigma_c\sqrt{1-\Delta}, \Lambda\bar{\gamma} + \mu(\bar{\gamma})\sigma_c\sqrt{1-\Delta}\right],$$

for any $c^\star$ *such that* $\mathcal{S}^{\mathrm{RN}}(c^\star, \bar{\gamma})$ *is nonempty.*

Proposition 4 extends the results of Andrews, Gentzkow, and Shapiro (2017) to the case where $\hat{\gamma}$ need not be a vector of estimation moments, and thus we may have $\Delta < 1$. It likewise extends Proposition 2. The resulting set of first-order asymptotic biases for $\hat{c}$ is centered at $\Lambda\bar{\gamma}$ with width proportional to $\sqrt{1-\Delta}$.

Unlike in Proposition 2, the degree of misspecification now enters the width through $\mu(\bar{\gamma}) = \sqrt{\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}}$. Intuitively, $\mu(\bar{\gamma})$ measures the degree of excess misspecification beyond $\sqrt{\bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}}$, which is the minimum necessary to allow $\bar{\gamma}(\tilde{S}) - \bar{\gamma}(S) = \bar{\gamma}$. If the degree of excess misspecification is small, then the first-order asymptotic bias of $\hat{c}$ is close to $\Lambda\bar{\gamma}$, while if the degree of excess misspecification is large, then a wider range of biases is possible.

PROOF OF PROPOSITION 4: The proof is similar to that for Proposition 2 in the main text. By Lemma 1, we again have

$$c^\star(h) = E_{F_0}\left[\phi_c(D_i)s_h(D_i)\right].$$

Note, next, that by the definition of $\mathcal{S}^{\mathrm{RN}}(c^\star, \bar{\gamma})$ and Lemma 1, for any $S \in \mathcal{S}^{\mathrm{RN}}(c^\star, \bar{\gamma})$ there exist $(h, z) \in \mathcal{H} \times \mathcal{Z}$ with $S = S(h, z)$, $c^\star(h) = c^\star$, and

$$E_{F_0}\left[\phi_\gamma(D_i)\left(s_h(D_i) + s_z(D_i)\right)\right] - E_{F_0}\left[\phi_\gamma(D_i)s_h(D_i)\right] = E_{F_0}\left[\phi_\gamma(D_i)s_z(D_i)\right] = \bar{\gamma}.$$

Thus, writing $\bar{\gamma}_z = E_{F_0}[\phi_\gamma(D_i)s_z(D_i)]$ and $\bar{c}_z = E_{F_0}[\phi_c(D_i)s_z(D_i)]$ for brevity, our task reduces to showing that

$$\left\{\bar{c}_z : z \in \mathcal{Z}, \bar{\gamma}_z = \bar{\gamma}, E_{F_0}\left[s_z(D_i)^2\right] \leq \mu^2\right\} = \left[\Lambda\bar{\gamma} - \mu(\bar{\gamma})\sigma_c\sqrt{1-\Delta}, \Lambda\bar{\gamma} + \mu(\bar{\gamma})\sigma_c\sqrt{1-\Delta}\right].$$

Define $s(D_i; \bar{\gamma}) = \phi_\gamma(D_i)'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}$, and

$$\varepsilon_z(D_i) = s_z(D_i) - s(D_i; \bar{\gamma}_z).$$

Note that $E_{F_0}[\phi_\gamma(D_i)\varepsilon_z(D_i)] = 0$ and $E_{F_0}[s(D_i; \bar{\gamma}_z)\varepsilon_z(D_i)] = 0$ by construction. We can write

$$\begin{aligned}
\bar{c}_z &= E_{F_0}\left[\phi_c(D_i)s_z(D_i)\right] \\
&= E_{F_0}\left[\phi_c(D_i)\phi_\gamma(D_i)'\right]\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}_z + E_{F_0}\left[\phi_c(D_i)\varepsilon_z(D_i)\right] \\
&= \Lambda\bar{\gamma}_z + E_{F_0}\left[\phi_c(D_i)\varepsilon_z(D_i)\right].
\end{aligned}$$

Next, define

$$\tilde{\phi}_c(D_i) = \phi_c(D_i) - \Lambda\phi_\gamma(D_i)$$

and note that

$$E_{F_0}\left[\phi_c(D_i)\varepsilon_z(D_i)\right] = E_{F_0}\left[\tilde{\phi}_c(D_i)\varepsilon_z(D_i)\right].$$

The Cauchy–Schwarz inequality then implies that

$$\begin{aligned}\left|E_{F_0}\left[\tilde{\phi}_c(D_i)\varepsilon_z(D_i)\right]\right| &\le \sqrt{E_{F_0}\left[\tilde{\phi}_c(D_i)^2\right]}\sqrt{E_{F_0}\left[\varepsilon_z(D_i)^2\right]} \\ &= \sqrt{\sigma_c^2 - \Lambda\Sigma_{\gamma\gamma}\Lambda'}\sqrt{E_{F_0}\left[\varepsilon_z(D_i)^2\right]} \\ &= \sigma_c\sqrt{1-\Delta}\sqrt{E_{F_0}\left[s_z(D_i)^2\right] - \bar{\gamma}_z'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}_z}.\end{aligned}$$

Combining these results, we see that for z such that $\bar{\gamma}_z = \bar{\gamma}$ and $E_{F_0}[s_z(D_i)^2] \le \mu^2$,

$$\bar{c}_z \in \left[\Lambda\bar{\gamma} - \sigma_c\sqrt{1-\Delta}\sqrt{\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}}, \Lambda\bar{\gamma} + \sigma_c\sqrt{1-\Delta}\sqrt{\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}}\right],$$

which are the bounds stated in the proposition. In particular,

$$0 \le E_{F_0}\left[\varepsilon_z(D_i)^2\right] \le \mu^2 - \bar{\gamma}_z'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}_z,$$

so if $\bar{\gamma}_z = \bar{\gamma}$, we must have $\bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma} \le \mu^2$ in order that $E_{F_0}[s_z(D_i)^2] \le \mu^2$. Hence, if $\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma} < 0$, there exists no z with $\bar{\gamma}_z = \bar{\gamma}$ and $E_{F_0}[s_z(D_i)^2] \le \mu^2$.

To complete the proof, it remains to show that these bounds are tight, so that for any $(\bar{c}, \bar{\gamma}, \mu)$ with

$$\bar{c} \in \left[\Lambda\bar{\gamma} - \sigma_c\sqrt{1-\Delta}\sqrt{\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}}, \Lambda\bar{\gamma} + \sigma_c\sqrt{1-\Delta}\sqrt{\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}}\right], \tag{16}$$

there exists $z \in \mathcal{Z}$ with $\bar{c}_z = \bar{c}$, $\bar{\gamma}_z = \bar{\gamma}$, and $E_{F_0}[s_z(D_i)^2] \le \mu^2$. If $\Delta < 1$, define

$$s^*(D_i; \bar{c}, \bar{\gamma}) = s(D_i; \bar{\gamma}) + \tilde{\phi}_c(D_i)\frac{\bar{c} - \Lambda\bar{\gamma}}{\sigma_c^2(1-\Delta)}.$$

Note that

$$E_{F_0}\left[\phi_\gamma(D_i)s^*(D_i; \bar{c}, \bar{\gamma})\right] = \bar{\gamma},$$

while

$$E_{F_0}\left[\phi_c(D_i)s^*(D_i; \bar{c}, \bar{\gamma})\right] = \Lambda\bar{\gamma} + E_{F_0}\left[\tilde{\phi}_c(D_i)^2\right]\frac{\bar{c} - \Lambda\bar{\gamma}}{\sigma_c^2(1-\Delta)} = \bar{c}.$$

Moreover,

$$\begin{aligned}E_{F_0}\left[s^*(D_i; \bar{c}, \bar{\gamma})^2\right] &= E_{F_0}\left[s(D_i; \bar{\gamma})^2\right] + E_{F_0}\left[\tilde{\phi}_c(D_i)^2\right]\frac{(\bar{c} - \Lambda\bar{\gamma})^2}{\sigma_c^4(1-\Delta)^2} \\ &= \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma} + \frac{(\bar{c} - \Lambda\bar{\gamma})^2}{\sigma_c^2(1-\Delta)}.\end{aligned}$$

However, by (16), we know that

$$|\bar{c} - \Lambda\bar{\gamma}| \le \sigma_c\sqrt{1-\Delta}\sqrt{\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}}$$

and thus that

$$\frac{(\bar{c}-\Lambda\bar{\gamma})^2}{\sigma_c^2(1-\Delta)} \leq \left(\mu^2 - \bar{\gamma}'\Sigma_{\gamma\gamma}^{-1}\bar{\gamma}\right),$$

so $E_{F_0}[s^*(D_i;\bar{c},\bar{\gamma})^2] \leq \mu^2$. By Assumption 4, however, there exists $z \in \mathcal{Z}$ with

$$E_{F_0}\left[\left(s_z(D_i) - s^*(D_i;\bar{c},\bar{\gamma})\right)^2\right] = 0,$$

and thus z yields $\bar{c}_z = \bar{c}$, $\bar{\gamma}_z = \bar{\gamma}$, and $E_{F_0}[s_z(D_i)^2] \leq \mu^2$ as desired. In cases with $\Delta = 1$, on the other hand, we can use $s^*(D_i;\bar{c},\bar{\gamma}) = s(D_i;\bar{\gamma})$. *Q.E.D.*

APPENDIX B: ASYMPTOTIC DIVERGENCE

This section studies the asymptotic behavior of the divergence

$$r_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right) = E_{F_{h,z}(t_h,0)}\left[\psi\left(\frac{f_{h,z}\left(D_i;\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)}{f_{h,z}\left(D_i;\frac{1}{\sqrt{n}},0\right)}\right)\right] \tag{17}$$

as $n \to \infty$, where, as in the main text, we assume that $\psi(1) = 0$ and $\psi''(1) = 2$. To derive our results, we impose the following assumption.

ASSUMPTION 6: *For $t = (t_h, t_z) \in \mathbb{R}^2$ and $f_{h,z}(D_i;t) = f_{h,z}(D_i;t_h,t_z)$, $f_{h,z}(D_i;t)$ is twice continuously differentiable in t at* 0, *and there exists an open neighborhood $\mathcal{B}$ of zero such that*

$$E_{F_0}\left[\sup_{t\in\mathcal{B}}\left(\left|\frac{\partial}{\partial t_z}f_{h,z}(D_i;t)\right| + \left|\frac{\partial^2}{\partial t_z^2}f_{h,z}(D_i;t)\right| \right.\right.$$
$$\left.\left. + \left|\frac{f_{h,z}(D_i;t_h,0)}{f_{h,z}(D_i;0)}\psi'\left(\frac{f_{h,z}(D_i;t)}{f_{h,z}(D_i;t)}\right)\frac{\frac{\partial}{\partial t_z}f_{h,z}(D_i;t)}{f_{h,z}(D_i;t)}\right|\right)\right],$$

$$E_{F_0}\left[\sup_{(t,\tilde{t})\in\mathcal{B}^2}\left|\frac{f_{h,z}(D_i;t_h,0)}{f_{h,z}(D_i;0)}\psi'\left(\frac{f_{h,z}(D_i;\tilde{t})}{f_{h,z}(D_i;t)}\right)\frac{\frac{\partial^2}{\partial t_z^2}f_{h,z}(D_i;\tilde{t})}{f_{h,z}(D_i;t)}\right|\right],$$

and

$$E_{F_0}\left[\sup_{(t,\tilde{t})\in\mathcal{B}^2}\left|\frac{f_{h,z}(D_i;t_h,0)}{f_{h,z}(D_i;0)}\psi''\left(\frac{f_{h,z}(D_i;\tilde{t})}{f_{h,z}(D_i;t)}\right)\left(\frac{\frac{\partial}{\partial t_z}f_{h,z}(D_i;\tilde{t})}{f_{h,z}(D_i;t)}\right)^2\right|\right]$$

are finite.

Under this assumption, we obtain the asymptotic approximation to divergence discussed in the main text.

PROPOSITION 5: *Under Assumptions* 3 *and* 6,

$$\lim_{n\to\infty} n \cdot r_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right) = E_{F_0}\left[s_z(D_i)^2\right].$$

PROOF OF PROPOSITION 5: Recall that $r_{h,z}(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ can be written as in (17). Assumption 6 and Leibniz's rule imply that for n sufficiently large, we can exchange integration and differentiation twice, so by Taylor's theorem with a mean-value residual,[1] we have that $n \cdot r_{h,z}(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ is equal to

$$n \cdot E_{F_0}\left[\begin{array}{c} \frac{f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;0)}\left(\psi\left(\frac{f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;t_n)}\right) + \psi'\left(\frac{f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;t_n)}\right)\frac{\frac{\partial}{\partial t_z} f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;t_n)}\frac{1}{\sqrt{n}}\right. \\ \left. + \frac{1}{2}\left(\psi'\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\frac{\frac{\partial^2}{\partial t_z^2} f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)} + \psi''\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\left(\frac{\frac{\partial}{\partial t_z} f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)^2\right)\frac{1}{n}\right) \end{array}\right]$$

for $t_n = (\frac{1}{\sqrt{n}}, 0)$, $\tilde{t}_n = (\frac{1}{\sqrt{n}}, \tilde{t}_{z,n})$, and $\tilde{t}_{z,n} \in [0, \frac{1}{\sqrt{n}}]$. Thus, since $\psi(1) = 0$ by assumption, we have that $n \cdot r_{h,z}(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ is equal to

$$E_{F_0}\left[\begin{array}{c} \sqrt{n}\psi'(1)\frac{\frac{\partial}{\partial t_z} f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;0)} \\ + \frac{1}{2}\frac{f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;0)}\left(\psi'\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\frac{\frac{\partial^2}{\partial t_z^2} f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right. \\ \left. + \psi''\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\left(\frac{\frac{\partial}{\partial t_z} f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)^2\right) \end{array}\right].$$

Assumption 6 and Leibniz's rule imply that for n sufficiently large,

$$\begin{aligned} E_{F_0}\left[\frac{\frac{\partial}{\partial t_z} f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;0)}\right] &= \int \frac{\partial}{\partial t_z} f_{h,z}\left(d; \frac{1}{\sqrt{n}}, 0\right) d\nu(d) \\ &= \frac{\partial}{\partial t_z}\int f_{h,z}\left(d; \frac{1}{\sqrt{n}}, 0\right) d\nu(d) = 0. \end{aligned}$$

Hence, we see that $n \cdot r_{h,z}\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)$ is equal to

$$E_{F_0}\left[\frac{1}{2}\frac{f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;0)}\left(\psi'\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\frac{\frac{\partial^2}{\partial t_z^2} f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right.\right.$$
$$\left.\left. + \psi''\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\left(\frac{\frac{\partial}{\partial t_z} f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)^2\right)\right].$$

[1]Specifically, note that for $q(t_h, t_z) = r_{h,z}(t_h, t_z)$, we can write

$$q(t_h, t_z) = q(t_h, 0) + \frac{\partial}{\partial t_z} q(t_h, 0)t_z + \frac{1}{2}\frac{\partial^2}{\partial t_z^2} q(t_h, \tilde{t}_z)t_z^2$$

with $\tilde{t}_z \in [0, t_z]$.

Since $\psi''(1)=2$, the dominated convergence theorem and Assumption 6 imply that

$$E_{F_0}\left[\frac{1}{2}\frac{f_{h,z}(D_i;t_n)}{f_{h,z}(D_i;0)}\left(\psi'\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\frac{\frac{\partial^2}{\partial t_z^2}f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right.\right.$$
$$\left.\left.+\psi''\left(\frac{f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)\left(\frac{\frac{\partial}{\partial t_z}f_{h,z}(D_i;\tilde{t}_n)}{f_{h,z}(D_i;t_n)}\right)^2\right)\right]$$
$$\to\frac{1}{2}E_{F_0}\left[\psi'(1)\frac{\frac{\partial^2}{\partial t_z^2}f_{h,z}(D_i;0)}{f_{h,z}(D_i;0)}+\psi''(1)\left(\frac{\frac{\partial}{\partial t_z}f_{h,z}(D_i;0)}{f_{h,z}(D_i;0)}\right)^2\right]$$
$$=E_{F_0}\left[\frac{1}{2}\psi'(1)\frac{\frac{\partial^2}{\partial t_z^2}f_{h,z}(D_i;0)}{f_{h,z}(D_i;0)}+s_z(D_i)^2\right].$$

However, Assumption 6 and Leibniz's rule imply that

$$E_{F_0}\left[\frac{\frac{\partial^2}{\partial t_z^2}f_{h,z}(D_i;0)}{f_{h,z}(D_i;0)}\right]=\int\frac{\partial^2}{\partial t_z^2}f_{h,z}(d;0)\,d\nu(d)=\frac{\partial^2}{\partial t_z^2}\int f_{h,z}(d;0)\,d\nu(d)=0,$$

so

$$\lim_{n\to\infty}n\cdot r_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)=E_{F_0}\left[s_z(D_i)^2\right],$$

as we wanted to show. *Q.E.D.*

APPENDIX C: ASYMPTOTIC DISTINGUISHABILITY

In Section 4.3 of the paper, and Section B above, we discuss that the neighborhoods studied in our local asymptotic analysis correspond to bounds on the asymptotic Cressie–Read divergence between $F_{h,z}(\frac{1}{\sqrt{n}},0)$ and $F_{h,z}(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}})$. In this section, we show that they also correspond to bounds on the asymptotic power of tests to distinguish $S(h,z)$ and $S(h,0)$.

PROPOSITION 6: *Under Assumption* 3, *the most powerful level-α test of the null hypothesis*

$$H_0:(D_1,\dots,D_n)\sim\bigtimes_{i=1}^{n}F_{h,z}\left(\frac{1}{\sqrt{n}},0\right)$$

against

$$H_1:(D_1,\dots,D_n)\sim\bigtimes_{i=1}^{n}F_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)$$

has power converging to $1-F_{N(0,1)}(v_\alpha-\sqrt{E_{F_0}[s_z(D_i)^2]})$ *for* v_α *the* $1-\alpha$ *quantile of the standard normal distribution*.

The proof of Proposition 6 shows that the most powerful test corresponds asymptotically to a z-test, where the z-statistic has mean $\sqrt{E_{F_0}[s_z(D_i)^2]}$ under H_1.

PROOF OF PROPOSITION 6: By the Neyman–Pearson lemma (see Theorem 3.2.1 in Lehmann and Romano (2005)), the most powerful level-α test of $H_0 : (D_1, \dots, D_n) \sim \bigtimes_{i=1}^n F_{h,z}(\frac{1}{\sqrt{n}}, 0)$ against $H_1 : (D_1, \dots, D_n) \sim \bigtimes_{i=1}^n F_{h,z}(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})$ rejects when the log likelihood ratio

$$\log\left(dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \Big/ dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, 0\right)\right)$$

exceeds a critical value $v_{\alpha,n}$ chosen to ensure rejection probability α under H_0 (and may randomize when the log likelihood ratio exactly equals the critical value). Here we again abbreviate $\bigtimes_{i=1}^n F = F^n$.

By Assumption 3 and the quadratic expansion of the likelihood in the proof of Lemma 1, however, we see that under $S(0, 0)$, for $g(D_i; h, z) = s_h(D_i) + s_z(D_i)$,

$$\begin{pmatrix} \log\left(\dfrac{dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, 0\right)}{dF_0^n}\right) & \log\left(\dfrac{dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)}{dF_0^n}\right)\end{pmatrix}' \\ \to_d N\left(\begin{pmatrix} -\frac{1}{2}E_{F_0}\left[g(D_i; h, 0)^2\right] \\ -\frac{1}{2}E_{F_0}\left[g(D_i; h, z)^2\right]\end{pmatrix}, \tilde{\Sigma}\right)$$

for

$$\tilde{\Sigma} = \begin{pmatrix} E_{F_0}\left[g(D_i; h, 0)^2\right] & E_{F_0}\left[g(D_i; h, 0)g(D_i; h, z)\right] \\ E_{F_0}\left[g(D_i; h, 0)g(D_i; h, z)\right] & E_{F_0}\left[g(D_i; h, z)^2\right]\end{pmatrix}.$$

Le Cam's third lemma thus implies that under $S(h, 0)$,

$$\begin{pmatrix} \log\left(\dfrac{dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, 0\right)}{dF_0^n}\right) & \log\left(\dfrac{dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)}{dF_0^n}\right)\end{pmatrix}' \\ \to_d N\left(\begin{pmatrix} \frac{1}{2}E_{F_0}\left[g(D_i; h, 0)^2\right] \\ -\frac{1}{2}E_{F_0}\left[g(D_i; h, z)^2\right] + E_{F_0}\left[g(D_i; h, 0)g(D_i; h, z)\right]\end{pmatrix}, \tilde{\Sigma}\right),$$

while under $S(h, z)$,

$$\begin{pmatrix} \log\left(\dfrac{dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, 0\right)}{dF_0^n}\right) & \log\left(\dfrac{dF_{h,z}^n\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right)}{dF_0^n}\right)\end{pmatrix}' \\ \to_d N\left(\begin{pmatrix} -\frac{1}{2}E_{F_0}\left[g(D_i; h, 0)^2\right] + E_{F_0}\left[g(D_i; h, 0)g(D_i; h, z)\right] \\ \frac{1}{2}E_{F_0}\left[g(D_i; h, z)^2\right]\end{pmatrix}, \tilde{\Sigma}\right).$$

Since

$$\log\left(\frac{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)}{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},0\right)}\right)=\log\left(\frac{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)}{dF^n_0}\right)-\log\left(\frac{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},0\right)}{dF^n_0}\right),$$

and $s_z(d)=s_h(d)=0$ when $h=z=0$, $g(D_i;h,0)-g(D_i;h,z)=-g(D_i;0,z)$, we see that

$$\log\left(\frac{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)}{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},0\right)}\right)$$
$$\to_d\begin{cases}N\left(-\frac{1}{2}E_{F_0}\left[g(D_i;0,z)^2\right],E_{F_0}\left[g(D_i;0,z)^2\right]\right) & \text{under } S(h,0),\\ N\left(\frac{1}{2}E_{F_0}\left[g(D_i;0,z)^2\right],E_{F_0}\left[g(D_i;0,z)^2\right]\right) & \text{under } S(h,z).\end{cases}$$

Hence, since $E_{F_0}[g(D_i;0,z)^2]=E_{F_0}[s_z(D_i)^2]$ and $v_{\alpha,n}$ corresponds to the $1-\alpha$ quantile of the log likelihood ratio under the null, we have that

$$\frac{\log\left(\frac{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}\right)}{dF^n_{h,z}\left(\frac{1}{\sqrt{n}},0\right)}\right)-v_{\alpha,n}}{\sqrt{E_{F_0}\left[s_z(D_i)^2\right]}}\to_d\begin{cases}N(-v_\alpha,1) & \text{under } S(h,0),\\ N\left(\sqrt{E_{F_0}\left[s_z(D_i)^2\right]}-v_\alpha,1\right) & \text{under } S(h,z),\end{cases}$$

for v_α the $1-\alpha$ quantile of a standard normal distribution, from which the result follows. *Q.E.D.*

APPENDIX D: NON-LOCAL MISSPECIFICATION

This section develops our informativeness measure based on probability limits, rather than first-order asymptotic bias.

Under Assumptions 1, 3, and 4, provided the estimators $\hat{c}$ and $\hat{\gamma}$ are regular in the sense discussed in Newey (1994), Theorem 2.1 of Newey (1994) implies that the probability limits $\tilde{c}(\cdot)$ and $\gamma(\cdot)$ are asymptotically linear functionals, in the sense that

$$\begin{aligned}&\lim_{t_z\to 0}\left\|\tilde{c}\left(F_{0,z}(0,t_z)\right)-c(\eta_0)-t_zE_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right\|/t_z=0 \quad \text{for all } z\in\mathcal{Z},\\ &\lim_{t_z\to 0}\left\|\gamma\left(F_{0,z}(0,t_z)\right)-\gamma(F_0)-t_zE_{F_0}\left[s_z(D_i)\phi_\gamma(D_i)\right]\right\|/t_z=0 \quad \text{for all } z\in\mathcal{Z}.\end{aligned}\tag{18}$$

Assumption 2 would be implied by an assumption that $(\hat{c},\hat{\gamma})$ are regular in the base model, so the assumption of regularity of $(\hat{c},\hat{\gamma})$ in the nesting model can be understood as a strengthening of Assumption 2. See Newey (1994) and Rieder (1994) for discussion.

Since (18) only restricts behavior as $t_z \to 0$ for fixed z, rather than studying $\tilde{\Delta}(\bar{r})$ as defined in the main text let us instead consider an analogue defined using finite collections of paths. Specifically, continuing to define $r_{h,z}(t_h, t_z) = E_{F_{h,z}(t_h,0)}[\psi(\frac{f_{h,z}(D_i;t_h,t_z)}{f_{h,z}(D_i;t_h,0)})]$, for each $z \in \mathcal{Z}$ let

$$\bar{t}(z,\mu) = \inf\{t_z \in \mathbb{R}_+ : r_{0,z}(0,t_z) \geq \mu\}$$

denote the largest value of t such that $r_{0,z}(0,t_z) < \mu$ for all $t_z < \bar{t}(z,\mu)$. Let $\mathcal{Z}_+ \subset \mathcal{Z}$ denote the set of $z \in \mathcal{Z}$ with $E_{F_0}[s_z(D_i)^2] > 0$.

Let $Q \subset \mathcal{Z}_+$ denote a finite subset of $\mathcal{Z}_+$, and let $\mathcal{Q}$ denote the set of all such finite subsets. Finally, let

$$\tilde{b}_N(\mu, Q) = \sup\{|\tilde{c}(F_{0,z}(0,t_z)) - c(\eta_0)| : z \in Q, t_z < \bar{t}(z,\mu)\}$$

denote the analogue of $\tilde{b}_N(\mu)$ based on the finite set of paths Q, and for $\varepsilon > 0$ let $\tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q)$, defined as

$$\sup\{|\tilde{c}(F_{0,z}(0,t_z)) - c(\eta_0)| : z \in Q, t_z < \bar{t}(z,\mu), \|\gamma(F_{0,z}(0,t_z)) - \gamma(F_0)\| \leq \varepsilon\sqrt{\mu}\},$$

denote the analogue of $\tilde{b}_{\mathrm{RN}}(\mu, Q)$ based on Q which allows the probability limit of $\hat{\gamma}$ to change by at most $\varepsilon\sqrt{\mu}$. Because $\tilde{b}_{\mathrm{RN},0}(\mu, Q)$ may equal 0 even for large μ due to the approximation error in (18), we consider limits as $\varepsilon \downarrow 0$ (i.e., as $\varepsilon \to 0$ from above). Based on these objects, we define the analogue of $\tilde{\Delta}(\mu)$ as

$$\tilde{\Delta}(\mu, \mathcal{Q}) = \sup_{Q_1 \in \mathcal{Q}} \inf_{Q_2 \in \mathcal{Q}} \lim_{\varepsilon \downarrow 0} \frac{\tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q_1)}{\tilde{b}_N(\mu, Q_2)},$$

provided the limit exists.

PROPOSITION 7: *Suppose Assumptions* 1, 3, *and* 4 *hold*, *that the estimators $\hat{c}$ and $\hat{\gamma}$ are regular, and that Assumption* 6 *holds for $h = 0$ and all $z \in \mathcal{Z}_+$. For $\psi(\cdot)$ twice continuously differentiable and* $\psi(1) = 0$, $\psi''(1) = 2$,

$$\sup_{Q_1 \in \mathcal{Q}} \inf_{Q_2 \in \mathcal{Q}} \lim_{\varepsilon \downarrow 0} \lim_{\mu \downarrow 0} \frac{\tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q_1)}{\tilde{b}_N(\mu, Q_2)} = \sqrt{1 - \Delta}.$$

It is important that we take the limit as $\mu \downarrow 0$ inside the limit as $\varepsilon \downarrow 0$ and the sup and inf, since this order of limits allows us to take advantage of the approximation result (18).

PROOF OF PROPOSITION 7: Note, first, that our Assumptions 1, 3, and 4 imply the conditions of Theorem 2.1 of Newey (1994) other than regularity of $(\hat{c}, \hat{\gamma})$. Specifically, conditions (i) and (ii) of Theorem 2.1 in Newey (1994) follow from our Assumptions 3 and 4. Condition (iii) is implied by our Assumption 1. Regularity of $(\hat{c}, \hat{\gamma})$ is assumed, so Theorem 2.1 of Newey (1994) implies (18).

Note, next, that for any $z \in \mathcal{Z}_+$, the proof of Proposition 5 implies that

$$\lim_{t_z \downarrow 0} r_{0,z}(0,t_z)/t_z^2 = E_{F_0}[s_z(D_i)^2].$$

Hence, as $\mu \downarrow 0$, $\bar{t}(z,\mu)/\sqrt{\mu} \to E[s_z(D_i)^2]^{-\frac{1}{2}}$. For all $z \in \mathcal{Z}_+$, (18) implies that

$$\lim_{\mu\downarrow 0} \sup_{t_z \leq \bar{t}(z,\mu)} \left\| \tilde{c}\big(F_{0,z}(0,t_z)\big) - c(\eta_0) - t_z E_{F_0}\big[s_z(D_i)\phi_c(D_i)\big] \right\| / t_z = 0,$$

$$\lim_{\mu\downarrow 0} \sup_{t_z \leq \bar{t}(z,\mu)} \left\| \gamma\big(F_{0,z}(0,t_z)\big) - \gamma(F_0) - t_z E_{F_0}\big[s_z(D_i)\phi_\gamma(D_i)\big] \right\| / t_z = 0,$$

and thus that

$$\left\{ \frac{1}{\sqrt{\mu}} \big(\tilde{c}\big(F_{0,z}(0,t_z)\big) - c(\eta_0), \gamma\big(F_{0,z}(0,t_z)\big) - \gamma(F_0) \big) : t_z \leq \bar{t}(z,\mu) \right\}$$
$$\to \left\{ \tilde{t}_z \big(E_{F_0}\big[s_z(D_i)\phi_c(D_i)\big], E_{F_0}\big[s_z(D_i)\phi_\gamma(D_i)\big] \big) : \tilde{t}_z \leq E_{F_0}\big[s_z(D_i)^2\big]^{-\frac{1}{2}} \right\}$$

in the Hausdorff sense as $\mu \downarrow 0$. Correspondingly, for any $Q \in \mathcal{Q}$,

$$\left\{ \frac{1}{\sqrt{\mu}} \big(\tilde{c}\big(F_{0,z}(0,t_z)\big) - c(\eta_0), \gamma\big(F_{0,z}(0,t_z)\big) - \gamma(F_0) \big) : z \in Q, t_z \leq \bar{t}(z,\mu) \right\}$$
$$\to \left\{ \tilde{t}_z \big(E_{F_0}\big[s_z(D_i)\phi_c(D_i)\big], E_{F_0}\big[s_z(D_i)\phi_\gamma(D_i)\big] \big) : z \in Q, \tilde{t}_z \leq E_{F_0}\big[s_z(D_i)^2\big]^{-\frac{1}{2}} \right\}.$$

Hence, for any nonempty $Q \in \mathcal{Q}$,

$$\frac{1}{\sqrt{\mu}} \tilde{b}_N(\mu, Q) \to \max \left\{ \frac{\left| E_{F_0}\big[s_z(D_i)\phi_c(D_i)\big] \right|}{E_{F_0}\big[s_z(D_i)^2\big]^{\frac{1}{2}}} : z \in Q \right\} \quad \text{as } \mu \downarrow 0.$$

Matters are somewhat more delicate for $\tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q)$. Note, in particular, that for $\varepsilon > 0$, as $\mu \downarrow 0$ we have

$$\begin{aligned}
&\frac{1}{\sqrt{\mu}} \tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q) \\
&\quad \to \sup \left\{ \tilde{t}_z E_{F_0}\big[s_z(D_i)\phi_c(D_i)\big] : z \in Q, \tilde{t}_z \leq E_{F_0}\big[s_z(D_i)^2\big]^{-\frac{1}{2}}, \tilde{t}_z \left\| E_{F_0}\big[s_z(D_i)\phi_\gamma(D_i)\big] \right\| \leq \varepsilon \right\} \\
&\quad = \sup \Bigg\{ \tilde{t}_z E_{F_0}\big[s_z(D_i)\phi_c(D_i)\big] : z \in Q, \\
&\qquad \tilde{t}_z \leq \min \left\{ E_{F_0}\big[s_z(D_i)^2\big]^{-\frac{1}{2}}, \frac{\varepsilon}{\left\| E_{F_0}\big[s_z(D_i)\phi_\gamma(D_i)\big] \right\|} \right\} \Bigg\},
\end{aligned}$$

where we define $\varepsilon/0 = \infty$ for $\varepsilon > 0$. Consequently,

$$\begin{aligned}
&\frac{1}{\sqrt{\mu}} \tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q) \\
&\quad \to \sup \Bigg\{ \tilde{t}_z \left| E_{F_0}\big[s_z(D_i)\phi_c(D_i)\big] \right| : z \in Q, \\
&\qquad \tilde{t}_z \leq \min \left\{ E_{F_0}\big[s_z(D_i)^2\big]^{-\frac{1}{2}}, \frac{\varepsilon}{\left\| E_{F_0}\big[s_z(D_i)\phi_\gamma(D_i)\big] \right\|} \right\} \Bigg\}.
\end{aligned}$$

Note, however, that by the Cauchy–Schwarz inequality and $E_{F_0}[s_z(D_i)^2] < \infty$, $E_{F_0}[s_z(D_i)\phi_c(D_i)]$ is finite for all $z \in \mathcal{Z}$, so for any z with $E_{F_0}[s_z(D_i)\phi_\gamma(D_i)] \neq 0$,

$$\frac{\varepsilon}{\left\| E_{F_0}\left[s_z(D_i)\phi_\gamma(D_i)\right]\right\|} E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right] \to 0$$

as $\varepsilon \downarrow 0$. Hence, as $\varepsilon \downarrow 0$,

$$\sup\left\{\tilde{t}_z\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right| : z \in Q, \tilde{t}_z \le \min\left\{E_{F_0}\left[s_z(D_i)^2\right]^{-\frac{1}{2}}, \frac{\varepsilon}{\left\|E_{F_0}\left[s_z(D_i)\phi_\gamma(D_i)\right]\right\|}\right\}\right\}$$
$$\to \max\left\{\frac{\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right|}{E_{F_0}\left[s_z(D_i)^2\right]^{\frac{1}{2}}} : z \in Q_0\right\}$$

for $Q_0 = \{z \in Q : E_{F_0}[s_z(D_i)\phi_\gamma(D_i)] = 0\}$, where we define this max to be zero if Q_0 is empty.

This immediately implies that

$$\lim_{\varepsilon\downarrow 0}\lim_{\mu\downarrow 0}\frac{\tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q_1)}{\tilde{b}_N(\mu, Q_2)} = \frac{\max\left\{\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right|/E_{F_0}\left[s_z(D_i)^2\right]^{\frac{1}{2}} : z \in Q_{1,0}\right\}}{\max\left\{\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right|/E_{F_0}\left[s_z(D_i)^2\right]^{\frac{1}{2}} : z \in Q_2\right\}}$$

for $Q_{1,0} = \{z \in Q_1 : E_{F_0}[s_z(D_i)\phi_\gamma(D_i)] = 0\}$, provided the denominator on the right-hand side is non-zero.[2]

To complete the proof, note that for $\mathcal{Q}_0$ the set of possible Q_0,

$$\sup_{Q_1\in\mathcal{Q}}\inf_{Q_2\in\mathcal{Q}}\lim_{\varepsilon\downarrow 0}\lim_{\mu\downarrow 0}\frac{\tilde{b}_{RN,\varepsilon}(\mu, Q_1)}{\tilde{b}_N(\mu, Q_2)} = \frac{\sup\limits_{Q_0\in\mathcal{Q}_0}\max\left\{\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right|/E_{F_0}\left[s_z(D_i)^2\right]^{\frac{1}{2}} : z \in Q_0\right\}}{\sup\limits_{Q\in\mathcal{Q}}\max\left\{\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right|/E_{F_0}\left[s_z(D_i)^2\right]^{\frac{1}{2}} : z \in Q\right\}}.$$

The proof of Proposition 2 shows, however, that

$$\max_{z\in\mathcal{Z}_+}\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right|/E_{F_0}\left[s_z(D_i)^2\right]^{\frac{1}{2}} = \sigma_c$$

and

$$\max_{z\in\mathcal{Z}_+:E_{F_0}[s_z(D_i)\phi_\gamma(D_i)]=0}\left|E_{F_0}\left[s_z(D_i)\phi_c(D_i)\right]\right|/E_{F_0}\left[s_z(D_i)^2\right]^{\frac{1}{2}} = \sigma_c\sqrt{1-\Delta}.$$

Hence,

$$\sup_{Q_1\in\mathcal{Q}}\inf_{Q_2\in\mathcal{Q}}\lim_{\varepsilon\downarrow 0}\lim_{\mu\downarrow 0}\frac{\tilde{b}_{\mathrm{RN},\varepsilon}(\mu, Q_1)}{\tilde{b}_N(\mu, Q_2)} = \sqrt{1-\Delta},$$

as we wanted to show. *Q.E.D.*

[2]If the denominator on the right-hand side is zero, we define the limit as $+\infty$.

APPENDIX E: ACCOUNTING FOR RICHER DEPENDENCE OF $\hat{c}$ ON THE DATA

In Section 5, for cases where the function $c(\theta)$ depends on the distribution of the data other than through θ, we effectively fix the distribution of the data at the empirical distribution for the purposes of estimating Δ and Λ. Here we discuss how to allow for uncertainty about the distribution of data in a special case, and present corresponding calculations for our applications.

Suppose in particular that

$$\hat{c} = \frac{1}{n}\sum_i c(\hat{\theta}; D_i) \tag{19}$$

for some function $c(\cdot)$. In contrast to the setup in Section 5, here we allow that $\hat{c}$ depends on the data directly, and not only through the dependence of $\hat{c}$ on $\hat{\theta}$.

In this case, one can show that the recipe in Section 5 applies, with the modification that

$$\hat{\phi}_c(D_i) = c(\hat{\theta}; D_i) + \hat{\Lambda}_{cg}\phi_g(D_i; \hat{\theta}), \tag{20}$$

where $\phi_g(D_i; \hat{\theta})$ and $\hat{\Lambda}_{cg}$ are as defined in Section 5, and $\hat{C}$ in the definition of $\hat{\Lambda}_{cg}$ is now given by the gradient of $\frac{1}{n}\sum_i c(\theta; D_i)$ with respect to θ at $\hat{\theta}$.

The proof of this result, which we omit, proceeds by noting that we can augment the GMM parameter vector as (c, θ), and correspondingly augment the moment equation as $(c(\theta; D_i) - c, \phi_g(D_i; \theta))$, following which we can derive the estimated influence function for $\hat{c}$ as we would for any element of $\hat{\theta}$.

In the cases of Attanasio, Meghir, and Santiago (2012) and Gentzkow (2007), we can represent the calculation of $\hat{c}$ in the form given in (19) and thus calculate $\hat{\Delta}$ using the modified estimated influence function in (20). In the case of Attanasio, Meghir, and Santiago (2012), the estimates in Table I change from 0.283, 0.227, and 0.056, respectively, to 0.277, 0.221, and 0.055. In the case of Gentzkow (2007), the estimates in Table II change from 0.514, 0.009, and 0.503, respectively, to 0.517, 0.008, and 0.507.

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