Econometrica Supplementary Material

SUPPLEMENT TO "STABLE MATCHING IN LARGE ECONOMIES"

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S.1. ANALYSIS OF THE EXAMPLE IN SECTION 2

LET r BE THE NUMBER of workers with each of the two types who are matched to f. We consider the following cases:

1. Suppose $r > q/2$. For any such matching, at least one position is vacant at firm f' because f' has q positions, but strictly more than q workers are matched to f out of the total of $2q$ workers. Thus, such a matching is blocked by f' and a type-θ' worker who is currently matched to f.

2. Suppose $r < q/2$. Consider the following cases.

(a) Suppose that there exists a type-θ worker who is unmatched. Then such a matching is unstable because that worker and firm f' block it (note that f' prefers θ most).

(b) Suppose that there exists no type-θ worker who is unmatched. This implies that there exists a type-θ' worker who is unmatched (because there are $2q$ workers in total, but firm f is matched to strictly fewer than q workers by assumption, and f' can be matched to at most q workers in any individually rational matching). Then, since f is the most preferred by all θ workers, a θ' worker prefers f to $\emptyset$, and there is some vacancy at f because $r < q/2$, the matching is blocked by a coalition of a type-θ worker, a type-θ' worker, and f.

S.2. PRELIMINARIES FOR THE CONTINUUM ECONOMY MODEL

S.2.1. *Lattice Property*

LEMMA S1: *The partially ordered set $(\mathcal{X}, \sqsubset)$ is a complete lattice.*

PROOF: For any subset $\mathcal{Y} \subset \mathcal{X}$, define

$$\overline{Y}(E) := \sup\left\{\sum_i Y_i(E_i) | \{E_i\} \text{ is a finite partition of } E \text{ in } \Sigma \text{ and } \{Y_i\} \text{ is a finite collection of measures in } \mathcal{Y}, \forall i\right\}, \quad \forall E,$$

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and $\underline{Y}$ analogously (by replacing "sup" with "inf"). We prove the lemma by showing that $\overline{Y} = \sup \mathcal{Y} \in \mathcal{Y}$ and $\underline{Y} = \inf \mathcal{Y} \in \mathcal{X}$.

First of all, note that $\overline{Y}$ and $\underline{Y}$ are monotonic, that is, for any $E \subset D$, we have $\overline{Y}(D) \geq \overline{Y}(E)$ and $\underline{Y}(D) \geq \underline{Y}(E)$, whose proof is straightforward and thus omitted.

We next show that $\overline{Y}$ and $\underline{Y}$ are measures. We only prove the countable additivity of $\overline{Y}$, since the other properties are straightforward to prove and also since a similar argument applies to $\underline{Y}$. To this end, consider any countable collection $\{E_i\}$ of disjoint sets in Σ and let $D = \bigcup E_i$. We need to show that $\overline{Y}(D) = \sum_i \overline{Y}(E_i)$. For doing so, consider any finite partition $\{D_i\}$ of D and any finite collection of measures $\{Y_i\}$. Letting $E_{ij} = E_i \cap D_j$, for any i, the collection $\{E_{ij}\}_j$ is a finite partition of E_i in Σ. Thus, we have

$$\sum_i Y_i(D_i) = \sum_i \sum_j Y_i(E_{ij}) \leq \sum_i \overline{Y}(E_i).$$

Since this inequality holds for any finite partition $\{D_i\}$ of D and collection $\{Y_i\}$, we must have $\overline{Y}(D) \leq \sum_i \overline{Y}(E_i)$. To show that the reverse inequality also holds, for each E_i, we consider any finite partition $\{E_{ij}\}_j$ of E_i in Σ and collection of measures $\{Y_{ij}\}_j$ in $\mathcal{Y}$. We prove that $\overline{Y}(D) \geq \sum_i \sum_j Y_{ij}(E_{ij})$, which would imply $\overline{Y}(D) \geq \sum_i \overline{Y}(E_i)$ as desired since the partition $\{E_{ij}\}_j$ and collection $\{Y_{ij}\}_j$ are arbitrarily chosen for each i. Suppose not for contradiction. Then, we must have $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij})$ for some k. Letting $E := \bigcup_{i=1}^k (\bigcup_j E_{ij})$, this implies $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij}) \leq \overline{Y}(E)$, where the second inequality holds by the definition of $\overline{Y}$. This contradicts the monotonicity of $\overline{Y}$ since $E \subset D$.

We now show that $\overline{Y}$ and $\underline{Y}$ are the supremum and infimum of $\mathcal{Y}$, respectively. It is straightforward to check that for any $Y \in \mathcal{Y}$, $Y \sqsubset \overline{Y}$ and $\underline{Y} \sqsubset Y$. Consider any $X, X' \in \mathcal{X}$ such that, for all $Y \in \mathcal{Y}$, $Y \sqsubset X$ and $X' \sqsubset Y$. We show that $\overline{Y} \sqsubset X$ and $X' \sqsubset \underline{Y}$. First, if $\overline{Y} \not\sqsubset X$ to the contrary, then there must be some $E \in \Sigma$ such that $\overline{Y}(E) > X(E)$. This means there are a finite partition $\{E_i\}$ of E and a collection of measures $\{Y_i\}$ in $\mathcal{Y}$ such that $\overline{Y}(E) \geq \sum Y_i(E_i) > X(E) = \sum X(E_i)$. Thus, for at least one i, we have $Y_i(E_i) > X(E_i)$, which contradicts the assumption that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$. An analogous argument can be used to show $X' \sqsubset \underline{Y}$. *Q.E.D.*

S.2.2. *Equivalence to Group Stability*

We say that a matching M is **group stable** if Condition 1 of Definition 1 holds and,

2′ *There are no* $F' \subseteq F$ *and* $M'_{F'} \in \mathcal{X}^{|F'|}$ *such that* $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M)$ for all $f \in F'$.

This definition strengthens our stability concept because it requires that matching be immune to blocks by coalitions that potentially involve multiple firms. Such stability concepts with coalitional blocks have been analyzed by Sotomayor (1999), Echenique and Oviedo (2006), and Hatfield and Kominers (2017), among others.

In our context, a matching is stable if and only if it is group stable. To see this, note first that any group stable matching is stable, because if Condition 2 is violated by a firm f and M'_f, then Condition 2′ is violated by a singleton set $F' = \{f\}$ and $M'_{\{f\}}$. The converse also holds. To see why, note that if Condition 2′ is violated by $F' \subseteq F$ and $M'_{F'}$, then Condition 2′ is violated by any $f \in F'$ and M'_f because $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M)$, by assumption.[1]

[1]By requiring $M'_f \sqsubset D^{\preceq f}(M)$ for all $f \in F'$ in Condition 2′, our group stability concept implicitly assumes that workers who consider joining a blocking coalition with $f \in F'$ use the current matching $(M_{f'})_{f' \neq f}$ as a ref-

S.2.3. *Stability and Pareto Efficiency*

DEFINITION S1: A matching M is **Pareto efficient** if there is no matching $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$, and **weakly Pareto efficient** if there is no matching M' such that $M' \succ_F M$ and $M' \succ_\Theta M$.[2]

PROPOSITION S1: *Any stable matching is weakly Pareto efficient, and Pareto efficient if each C_f is a choice function.*

PROOF: Suppose that matching M is not weakly Pareto efficient. Then, by definition of weak Pareto efficiency, there exist M' and $f \in F$ such that $M' \succ_\Theta M$ and $M'_f \succ_f M_f$.

Next, since $M' \succ_\Theta M$, for each $\tilde{f}$, we have $D^{\succeq \tilde{f}}(M') \sqsupset D^{\succeq \tilde{f}}(M)$, or

$$\sum_{f':f' \succeq_P \tilde{f}} M'_{f'}(\Theta_P \cap E) \geq \sum_{f':f' \succeq_P \tilde{f}} M_{f'}(\Theta_P \cap E), \quad \forall E \in \Sigma.$$

This implies that

$$\sum_{f':f' \succeq_P f^P_-} M'_{f'}(\Theta_P \cap E) \geq \sum_{f':f' \succeq_P f^P_-} M_{f'}(\Theta_P \cap E), \quad \forall E \in \Sigma,$$

where f^P_- refers to the firm that is ranked immediately above f according to P (whenever it is well defined),[3] or equivalently

$$\sum_{f':f' \succ_P f} M'_{f'}(\Theta_P \cap E) \geq \sum_{f':f' \succ_P f} M_{f'}(\Theta_P \cap E), \quad \forall E \in \Sigma.$$

This in turn implies that, for each P,

$$\sum_{f':f' \preceq_P f} M'_{f'}(\Theta_P \cap E) \leq \sum_{f':f' \preceq_P f} M_{f'}(\Theta_P \cap E), \quad \forall E \in \Sigma,$$

or equivalently,

$$D^{\preceq f}\big(M'\big) \sqsubset D^{\preceq f}(M).$$

By definition, $M'_f \sqsubset D^{\preceq f}(M')$, so we have $M'_f \sqsubset D^{\preceq f}(M)$.

Collecting the observations so far, we conclude that f and M'_f block M, implying that M is not stable. We have thus established that stability implies weak Pareto efficiency.

Suppose now that each C_f is a choice function and that a stable matching M is not Pareto efficient. Then, there is another matching $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta$

erence point. This means that workers are available to firm f as long as they prefer f to their current matching. However, given that a more preferred firm $f' \in F'$ may be making offers to workers in $D^{\preceq f}(M)$ as well, the set of workers available to f may be smaller. Such a consideration would result in a weaker notion of group stability. Any such concept, however, will be equivalent to our notion of stability because this subsection establishes that even the most restrictive notion of group stability—the concept using $D^{\preceq f}(M)$ in Condition 2′—is equivalent to stability, while stability is weaker than any group stability concept described above.

[2]In the definition of Pareto efficiency, the condition that $M' \succeq_\Theta M$ and $M' \neq M$ implies that at least some workers are strictly better off under M' since workers have strict preferences, and hence M' Pareto dominates M (though all firms may be indifferent between M and M').

[3]This is defined later as an immediate predecessor. Formally, $f^P_- \succ_P f$ and if $f' \succ_P f$, then $f' \succeq_P f^P_-$.

M. Choose any firm $f \in F$ with $M_f \neq M'_f$ and note that since C_f is a choice function, we have $C_f(M_f \vee M'_f) = M'_f \neq M_f$, which means $M'_f \succ_f M_f$. Given this, a contradiction can be drawn following the same argument as above. *Q.E.D.*

S.3. EQUIVALENCE WITH WORKER-PROPOSING DA

In this section, we establish the equivalence between a repeated application of our fixed-point mapping and the worker-proposing DA process when firms have substitutable preferences. To do so, we assume that each firm's choice is always unique, that is, C_f is a choice function. Then, the substitutability of firm f's reference becomes

$$R_f(X) \sqsubset R_f(X') \quad \text{whenever } X \sqsubset X'. \tag{SUB}$$

Let $\hat{X}_f^t$ denote the cumulative measure of workers proposing to the firm f from round 1 through t of the worker-proposing DA process. Let $\hat{A}_f^t$ denote the measure of workers (tentatively) accepted by f in round t. Let $(\hat{X}_f^0, \hat{A}_f^0) = (\mathbf{0}, \mathbf{0})$. In the first round, all workers propose to their most preferred firms, which means that for any $P \in \mathcal{P}$ and $E \subset \Theta_P$,

$$\hat{X}_f^1(E) = \begin{cases} G(E), & \text{if } f \succ_P f', \forall f' \neq f, \\ 0, & \text{otherwise.} \end{cases} \tag{S1}$$

Given this,

$$\hat{A}_f^1 = C_f\big(\hat{X}_f^1\big). \tag{S2}$$

For $t \geq 2$, the pair $(\hat{X}_f^t, \hat{A}_f^t)$ is recursively defined as follows: For any $P \in \mathcal{P}$ and $E \subset \Theta_P$,

$$\hat{X}_f^t(E) = \begin{cases} G(E), & \text{if } f \succ_P f', \forall f' \neq f, \\ R_{f_-^P}\big(\hat{X}_{f_-^P}^{t-1} - \hat{X}_{f_-^P}^{t-2} + \hat{A}_{f_-^P}^{t-2}\big)(E) + \hat{X}_f^{t-1}(E), & \text{otherwise,} \end{cases} \tag{S3}$$

$$\hat{A}_f^t = C_f\big(\hat{X}_f^t - \hat{X}_f^{t-1} + \hat{A}_f^{t-1}\big). \tag{S4}$$

The first expression in (S3) is straightforward, given that all workers who most prefer f propose to f in the first round. To understand the second expression, the cumulative measure of workers proposing to f (which is not most preferred according to P) from round 1 through t is obtained by adding to $\hat{X}_f^{t-1}$—that is, measure of workers proposing to f from round 1 through $t-1$—the measure of workers who newly propose to f in round t. The latter workers then coincide with those rejected by f's immediate predecessor (i.e., f_-^P) in round $t-1$, whose measure is equal to $R_{f_-^P}(\hat{X}_{f_-^P}^{t-1} - \hat{X}_{f_-^P}^{t-2} + \hat{A}_{f_-^P}^{t-2})$. To see this, note that in round $t-1$, the firm f_-^P considers and accepts/rejects among those tentatively accepted by f_-^P in round $t-2$ (their measure being equal to $\hat{A}_{f_-^P}^{t-2}$) and those newly proposing to f_-^P in round $t-1$ (their measure being equal to $\hat{X}_{f_-^P}^{t-1} - \hat{X}_{f_-^P}^{t-2}$). The expression in (S4) can be understood similarly.

Let $\tilde{X}^0$ denote a profile of zero measures (i.e., the profile has one zero measure for each firm in $\tilde{F}$). Define iteratively $\tilde{X}^t = T(\tilde{X}^{t-1})$ for each $t \geq 1$, where T is our fixed-point mapping.

PROPOSITION S2: *If* (SUB) *holds for all* $f \in F$, *then* $\hat{X}_f^t = \tilde{X}_f^t$ *in each round* $t \geq 1$.

Before starting the proof, we establish the following lemma:

LEMMA S2: *Given RP*, (SUB) *is equivalent to the path independence*:

$$C_f(X') = C_f\big(C_f(X) + X' - X\big), \quad \forall X \sqsubset X'. \tag{PI}$$

PROOF: That (PI) implies (SUB) follows immediately from noting that

$$C_f(X') = C_f\big(C_f(X) + X' - X\big) \sqsubset C_f(X) + X' - X,$$

and thus $X - C_f(X) \sqsubset X' - C_f(X')$ or $R_f(X) \sqsubset R_f(X')$.

To prove the converse, for any subpopulations X and X' with $X \sqsubset X'$, let $Z = C_f(X) + X' - X$. Then, by SUB, we have $C_f(X') \sqsubset Z$. Since $Z \sqsubset X'$, RP implies $C_f(Z) = C_f(X')$, which is equivalent to (PI), as desired. *Q.E.D.*

PROOF OF PROPOSITION S2: We need to show that for each $s \geq 1$ and for each $P \in \mathcal{P}$ and $E \subset \Theta_P$,

$$\hat{X}_f^s(E) = \tilde{X}_f^s(E) = T_f\big(\tilde{X}^{s-1}\big)(E) = \begin{cases} G(E), & \text{if } f \succ_P f', \forall f' \neq f, \\ R_{f_-^P}\big(\tilde{X}_{f_-^P}^{s-1}\big)(E), & \text{otherwise.} \end{cases} \tag{S5}$$

Let us first establish that, for all $s \geq 1$, $\hat{A}_f^s = C_f(\hat{X}_f^s)$. This holds for $s = 1$ due to (S2). Assuming inductively that this holds for all $s \leq t-1$, we have

$$\hat{A}_f^t = C_f\big(\hat{X}_f^t - \hat{X}_f^{t-1} + \hat{A}_f^{t-1}\big) = C_f\big(\hat{X}_f^t - \hat{X}_f^{t-1} + C_f\big(\hat{X}_f^{t-1}\big)\big) = C_f\big(\hat{X}_f^t\big),$$

where the last equality holds due to (PI) and the fact that $\hat{X}_f^{t-1} \sqsubset \hat{X}_f^t$.

To show (S5), consider $s = 1$ and note that if f is not most preferred according to P, then

$$\tilde{X}_f^1(E) = T_f\big(\tilde{X}^0\big)(E) = R_{f_-^P}\big(\tilde{X}_{f_-^P}^0\big)(E) = R_{f_-^P}(\mathbf{0})(E) = 0,$$

while, if f is most preferred, then $\tilde{X}_f^1(E) = G(E)$. This means that $\tilde{X}_f^1$ coincides with $\hat{X}_f^1$ given in (S1), so (S5) holds for $s = 1$. Assume inductively that (S5) holds for all $s \leq t-1$. To show that it holds for $s = t$, for any $P \in \mathcal{P}$ and $E \subset \Theta_P$, letting $g = f_-^P$ (to simplify notation), we have

$$\begin{aligned}
\hat{X}_f^t(E) &= R_g\big(\hat{X}_g^{t-1} - \hat{X}_g^{t-2} + \hat{A}_g^{t-2}\big)(E) + \hat{X}_f^{t-1}(E) \\
&= R_g\big(\hat{X}_g^{t-1} - \hat{X}_g^{t-2} + C_g\big(\hat{X}_g^{t-2}\big)\big)(E) + \hat{X}_f^{t-1}(E) \\
&= \hat{X}_g^{t-1}(E) - \hat{X}_g^{t-2}(E) + C_g\big(\hat{X}_g^{t-2}\big)(E) \\
&\quad - C_g\big(\hat{X}_g^{t-1} - \hat{X}_g^{t-2} + C_g\big(\hat{X}_g^{t-2}\big)\big)(E) + \hat{X}_f^{t-1}(E) \\
&= \hat{X}_g^{t-1}(E) - \hat{X}_g^{t-2}(E) + C_g\big(\hat{X}_g^{t-2}\big)(E) - C_g\big(\hat{X}_g^{t-1}\big)(E) + \hat{X}_f^{t-1}(E) \\
&= R_g\big(\hat{X}_g^{t-1}\big)(E) - R_g\big(\hat{X}_g^{t-2}\big)(E) + \hat{X}_f^{t-1}(E) \\
&= R_g\big(\hat{X}_g^{t-1}\big)(E) - R_g\big(\hat{X}_g^{t-2}\big)(E) + R_g\big(\tilde{X}_g^{t-2}\big)(E) = R_g\big(\tilde{X}_g^{t-1}\big)(E)
\end{aligned}$$

as desired, where the fourth equality holds due to Lemma S2, while the last two equalities hold due to the inductive assumption that for all $s \le t-1$, $\hat{X}_f^s(E) = R_{f_-^P}(\tilde{X}_{f_-^P}^{s-1})(E)$ and $\hat{X}_g^s = \tilde{X}_g^s$. *Q.E.D.*

S.4. ANALYSIS OF THE EXAMPLES IN SECTION 4

S.4.1. *Example for Remark 2*

Let us modify Example 3 by assuming that f_1 has a "Leontief" preference and would like to hire mass $a < 1$ of type-θ workers per unit mass of type-θ' workers, while keeping preferences of all other players unchanged. Thus, f_1's choice function becomes

$$C_{f_1}(X_{f_1}) = \left(a \min\left\{\frac{x_1}{a}, x_1'\right\}, \min\left\{\frac{x_1}{a}, x_1'\right\}\right), \tag{S6}$$

where $X_{f_1} = (x_1, x_1')$ is the measures of type-θ and type-θ' workers available to f_1. As in Example 3, without loss, we can set $x_1 = G(\theta) = \frac{1}{2}$ and $x_2' = G(\theta') = \frac{1}{2}$, and consider $X = (\frac{1}{2}, x_1', x_2, \frac{1}{2})$ as our candidate measures. Using this with (6), (S6), and C_{f_2} in (3), the fixed-point mapping is given as follows: for any $X = (\frac{1}{2}, x_1', x_2, \frac{1}{2})$,

$$T_{f_1}(X) = \left(\frac{1}{2}, R_{f_2}\left(x_2, \frac{1}{2}\right)(\theta')\right) = \left(\frac{1}{2}, x_2\right), \tag{S7}$$

$$T_{f_2}(X) = \left(R_{f_1}\left(\frac{1}{2}, x_1'\right)(\theta), \frac{1}{2}\right) = \left(\frac{1}{2} - a x_1', \frac{1}{2}\right). \tag{S8}$$

Letting $\phi_1(x_2) = x_2$ and $\phi_2(x_1') = \frac{1}{2} - a x_1'$ and assuming $q \le \frac{1}{4}$, the mapping $(x_1', x_2) \mapsto (\phi_1(x_2), \phi_2(x_1'))$ can be depicted as in Figure S1. The unique fixed point of T is given as

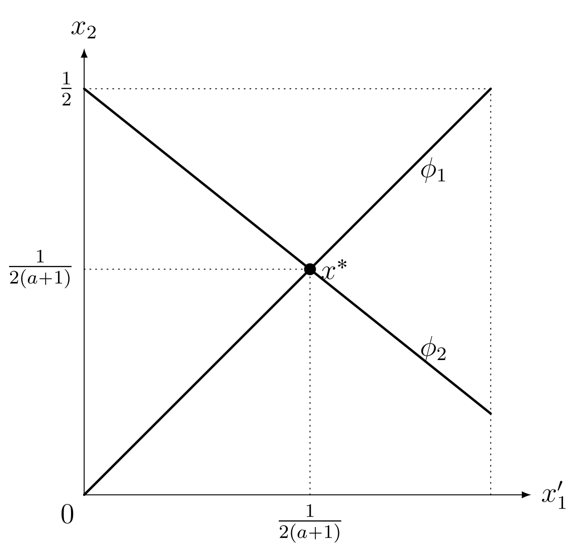

FIGURE S1.—Fixed point of mapping T.

$x_1' = x_2 = \frac{1}{2(a+1)}$, which yields the corresponding stable matching

$$M = \begin{pmatrix} f_1 & f_2 \\ \frac{a}{2(a+1)}\theta + \frac{1}{2(a+1)}\theta' & \frac{1}{2(a+1)}\theta + \frac{a}{2(a+1)}\theta' \end{pmatrix}.$$

To show that the tâtonnement process with any initial point converges to the fixed point, it suffices to show that $T^2 = T \circ T$ is a contraction mapping, and to invoke Proposition 1. To do so, consider any $X = (\frac{1}{2}, x_1', x_2, \frac{1}{2})$ and $Y = (\frac{1}{2}, y_1', y_2, \frac{1}{2})$. Then, $T^2(X) = (\frac{1}{2}, \frac{1}{2} - ax_1', \frac{1}{2} - ax_2, \frac{1}{2})$ and $T^2(Y) = (\frac{1}{2}, \frac{1}{2} - ay_1', \frac{1}{2} - ay_2, \frac{1}{2})$. Thus,

$$\|T^2(X) - T^2(Y)\| = \|(0, -a(x_1' - y_1'), a(x_2 - y_2), 0)\| = a\|X - Y\|,$$

which implies that T^2 is a contraction mapping, since $a < 1$.

S.4.2. *Analysis of Example 4*

Consider the following two cases:

1. Suppose f_1 hires measure 1/2 of each type of workers (i.e., all workers). In such a matching, none of the capacity of f_2 is filled. Thus, such a matching is blocked by f_2 and type-θ' workers (note that every type-θ' worker is currently matched with f_1, so they are willing to participate in the block).
2. Suppose f_1 hires no worker. Then, the only candidate for a stable matching is one in which f_2 hires measure 1/2 of the type-θ workers (otherwise f_2 and unmatched workers of type θ would block the matching). Then, because f_1 is the top choice of all type-θ workers and type-θ' workers prefer f_1 to $\emptyset$, the matching is blocked by a coalition of 1/2 of the type-θ workers, 1/2 of the type-θ' workers, and f_1.

S.4.3. *Analysis of Example 5*

Consider first a matching in which f_2 hires a positive mass of type-θ' workers. Then, it must hire type-θ' workers only and hire all of them. (Recall that type θ' prefers f_2 while f_2 has the capacity of 0.5.) Then, f_1 hires no one, implying that mass 0.6 of type-θ workers are all unmatched. Then, f_2 could form a blocking coalition with mass 0.6 of type-θ workers. Consider second a matching in which f_2 hires zero mass of type-θ' workers. Then, f_1 must hire the entire type-θ' workers and the same mass of type-θ workers (since all type-θ' workers are available and the type θ prefers f_1 to f_2). This would only leave the mass 0.2 of type-θ workers for f_2 to hire. Then, f_2 could form a blocking coalition with the mass 0.4 of type-θ' workers (since the type θ' prefers f_2 to f_1).

S.5. OMITTED EXAMPLES FROM SECTION 6

EXAMPLE S1—Substitutable Preference: Consider Example 3 again and assume that the preference of firm f_2 as well as that of workers remains the same, but f_1's preference is changed as follows: it has a capacity equal to 1 (which is large enough to hire the entire workers); for the first quarter of its capacity, it hires workers according to the responsive preference: $\theta \succ \theta'$; for the remaining capacity, it is indifferent to hiring any number of additional workers. The resulting choice correspondence is

$$C_{f_2}(x,x') = \begin{cases} \{(x,x')\}, & \text{if } x+x' \le \dfrac{1}{4}, \\ \{x\} \times \left[\dfrac{1}{4}-x, x'\right], & \text{if } x+x' > \dfrac{1}{4} \text{ and } x \le \dfrac{1}{4}, \\ \left[\dfrac{1}{4}, x\right] \times [0, x'], & \text{if } x+x' > \dfrac{1}{4} \text{ and } x > \dfrac{1}{4}. \end{cases}$$

One can verify that this preference is substitutable, and the set of stable matchings is

$$\mathcal{M}^* = \left\{ (x_i, x'_i)_{i=1,2} | x_1 \in \left[\frac{1}{4}, \frac{1}{2}\right], x'_1 \in \left[0, \frac{1}{2} - x_1\right], \text{ and } (x_2, x'_2) = \left(\frac{1}{2} - x_1, x_1\right) \right\}.$$

Observe M^* contains side-optimal matchings: the firm-optimal/worker-pessimal matching is $(x_1, x'_1) = (\frac{1}{4}, \frac{1}{4})$ and $(x_2, x'_2) = (\frac{1}{4}, \frac{1}{4})$, and the worker-optimal/firm-pessimal matching is $(x_1, x'_1) = (\frac{1}{2}, 0)$ and $(x_2, x'_2) = (0, \frac{1}{2})$. It can be seen easily, however, that M^* is not a lattice while C_{f_2} fails the strong-set monotonicity.

EXAMPLE S2—The Role of Order Continuity in Theorem 4(ii): Consider our leading example with two types of workers, each of mass $\frac{1}{2}$, with the same preferences as before. As before, the measures of available workers can be described succinctly by (x'_1, x_2), where x'_1 is the measure of type-θ' workers available to firm 1 and x_2 is the measure of type-θ workers available to firm 2. (As before, the measure of type-θ workers available to firm 1 and that of type-θ' workers available to firm 2 are always $\frac{1}{2}$.) Suppose firms' preferences are given by two choice functions:

$$C_{f_1}\left(\frac{1}{2}, x'_1\right) = \begin{cases} \left(\dfrac{1}{4}, x'_1\right), & \text{if } x'_1 \le \dfrac{1}{3}; \\ \left(\dfrac{1}{4} - \dfrac{1}{4}x'_1, x'_1\right), & \text{if } x'_1 > \dfrac{1}{3}; \end{cases}$$

and

$$C_{f_2}\left(x_2, \frac{1}{2}\right) = \begin{cases} \left(x_2, \dfrac{1}{4}\right), & \text{if } x_2 \le \dfrac{1}{3}; \\ \left(x_2, \dfrac{1}{4} - \dfrac{1}{4}x_2\right), & \text{if } x_2 > \dfrac{1}{3}, \end{cases}$$

where we set $x_1 = x'_2 = 1/2$ as in other examples. As can be seen, the choice function fails to be order-continuous. Letting $\phi_1(x_2) = R_{f_2}(x_2, \frac{1}{2})$ and $\phi_2(x'_1) = R_{f_1}(\frac{1}{2}, x'_1)$, Figure S2 depicts ϕ_1 and ϕ_2 in (x'_1, x_2) plane, whose intersection gives a fixed point of T. As can be seen, there exists a unique fixed point $(\frac{1}{4}; \frac{1}{4})$. Yet, if we iterate T from the largest point of the space $(\frac{1}{2}, \frac{1}{2})$, the algorithm gets "stuck" at $(\frac{1}{3}, \frac{1}{3}) = \lim_{k\to\infty} T^k(\frac{1}{2}, \frac{1}{2})$, which does not correspond to a stable matching.

EXAMPLE S3—The Role of LoAD for Theorem 6: Consider a continuum economy with worker types θ_1 and θ_2 (each with measure 1/2) and firms f_1 and f_2. Preferences are as follows:

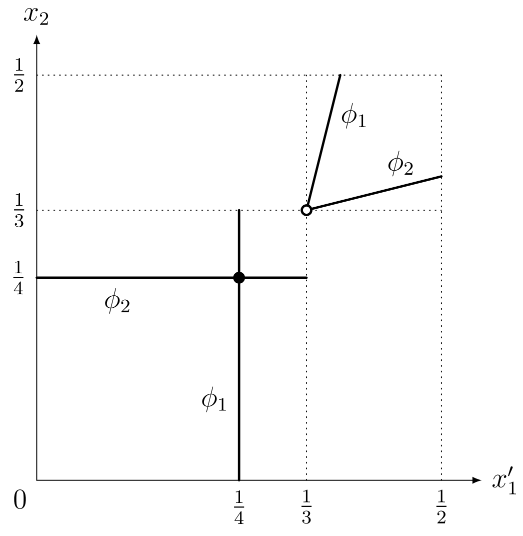

FIGURE S2.—Order continuity fails at $(x_1', x_2) = (1/3, 1/3)$.

1. Firm f_1 wants to hire as many workers of type θ_2 as possible if no worker of type θ_1 is available, but if any positive measure of type-θ_1 workers is available, then f_1 wants to hire only type-θ_1 workers and no type-θ_2 workers at all, and f_1 wants to hire only up to measure 1/3 of type-θ_1 workers.

2. The preference of firm f_2 is symmetric, changing the roles of worker types θ_1 and θ_2. More specifically, firm f_2 wants to hire as many workers of type θ_1 as possible if no worker of type θ_2 is available, but if any positive measure of type-θ_2 workers is available, then f_2 wants to hire only type-θ_2 workers and no type-θ_1 workers at all, and f_2 wants to hire only up to measure 1/3 of type-θ_2 workers.

3. Worker preferences are as follows:

$$\theta_1 : f_2 \succ f_1 \succ \emptyset,$$
$$\theta_2 : f_1 \succ f_2 \succ \emptyset.$$

Clearly, the firm preferences are substitutable. Note also that the worker-optimal stable matching is

$$\underline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta_2 & \frac{1}{2}\theta_1 \end{pmatrix},$$

where the notation is such that measure 1/2 of type-θ_1 workers are matched to f_2 and measure 1/2 of type-θ_2 workers are matched to f_1.[4] Given this, it is straightforward to check that the rich preferences hold.[5] Finally, firm preferences violate LoAD because, for instance, the choice of f_1 from measure 1/2 of θ_2 is to hire all of them, but even adding

[4]That this is a worker-optimal stable matching follows from the fact that the worker-proposing DA ends after the first round where each worker applies to and is accepted by her preferred firm.

[5]Under any matching $\hat{M} \neq \underline{M}$ that satisfies $\hat{M}_f = C_f(\hat{M}_f \vee \underline{M}_f)$ for all f, some firm, say f_1, must be matched with a positive measure of θ_1 workers. Given that $\hat{M}$ is individually rational, this implies that f_1 is not matched

a measure $\epsilon < 1/2$ of type-θ_1 workers would cause f_1 to reject all θ_2 workers. As it turns out, there is a firm-optimal stable matching that is different from $\underline{M}$ and given as follows:

$$\overline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{3}\theta_1 & \frac{1}{3}\theta_2 \end{pmatrix}.$$

S.6. ANALYSIS FOR SECTION 6

S.6.1. *Preliminary Analysis*

Throughout this section, we study the choice function of any individual firm with responsive preference while omitting the firm index from all notations for simplicity.

We begin by characterizing the choice function induced by the preferences. To this end, note first that, given a measure X of available workers, the quota constraint imposes the following constraint on any choice $X' \sqsubset X$:

$$X'(E) \leq \inf_{E' \subset E, E' \in \Sigma} X(E \setminus E') + \mathcal{Q}(\{t \in \mathcal{T} | E' \cap \Theta^t \neq \emptyset\}), \quad \forall E \in \Sigma. \tag{S9}$$

Then, the firm's optimization problem becomes

$$[P] \qquad \max_{X'} \int_0^1 s_f(\theta)\, dX'(\theta) \quad \text{subject to (S9)}.$$

We identify a (unique) solution to $[P]$ via *Greedy Algorithm* defined below, which consists of multiple steps at each of which the firm hires workers with highest scores (among remaining workers) until the quota constraint becomes binding for some subset of ethnic types.

GREEDY ALGORITHM—GA: Set $T_0 = \emptyset$. For each Step $k \geq 1$, define T_k as a maximal element (in the set inclusion sense) of

$$\begin{aligned} \arg \max_{T' \subset \mathcal{T} \setminus (\bigcup_{j=0}^{k-1} T_j)} \inf \Bigg\{ & s \in [0,1] | X(\{\theta \in \Theta | \tau(\theta) \in T' \text{ and } s_f(\theta) \in [s,1]\}) \\ & < \mathcal{Q}_f\left(\left(\bigcup_{j=0}^{k-1} T_j\right) \cup T'\right) - \mathcal{Q}_f\left(\bigcup_{j=0}^{k-1} T_j\right)\Bigg\}, \end{aligned} \tag{S10}$$

and s_k as the resulting maximum.[6] If $\bigcup_{j=1}^{k} T_j = \mathcal{T}$, stop; otherwise iterate to Step $k+1$.

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with any θ_2 workers. Also, since f_2 is matched with no more than measure 1/3 workers of θ_2 under any individual rational matching, at least measure 1/6 of θ_2 workers are unemployed under $\hat{M}$, which means that these workers belong to $\hat{M}^{f_2}_{\bar{F}}$ since they prefer f_2 to $\emptyset$ and $\emptyset \notin \bar{F}$. If they are available to f_2 in addition to $\underline{M}_{f_2}$, then f_2 would choose not to be matched with any θ_1 workers, to whom it is matched under $\underline{M}_{f_2}$. Thus, the rich preference condition is satisfied.

[6] We assume the infimum of an empty set is 1. Note that s_k is strictly decreasing in k since otherwise there would exist a k such that $s_k \geq s_{k-1}$ and

$$\bar{F}_{T_k}(s_{k-1}) + \bar{F}_{T_{k-1}}(s_{k-1}) \geq \bar{F}_{T_k}(s_k) + \bar{F}_{T_{k-1}}(s_{k-1}) = \mathcal{Q}\left(\left(\bigcup_{j=1}^{k-2} T_j\right) \cup T_k \cup T_{k-1}\right) - \mathcal{Q}\left(\bigcup_{j=1}^{k-2} T_j\right),$$

contradicting the fact that T_{k-1} is the maximal element of the maximizer in Step $k-1$.

Each step iteratively identifies the cutoff score for a group of workers whose residual quota is most binding. Let m denote the last step of this procedure, at which $\bigcup_{j=1}^{m} T_j = \mathcal{T}$.

Below, we first show that GA yields a unique profile $(s_k, T_k)_{k=1}^{m}$ (Lemma S4), and use this profile to identify a unique solution to $[P]$ (Proposition S3).

To begin, from any subpopulation X, one can obtain a corresponding score distribution for each ethnic type $t \in \mathcal{T}$, denoted F_t, as follows: for any (Borel) set $S \subset [0, 1]$,

$$F_t(S) = X\big(\{\theta \in \Theta^t | s(\theta) \in S\}\big).$$

By abuse of notation, we denote, for each $s \in [0, 1]$,

$$F_t(s) = F_t\big([0, s]\big) \quad \text{and} \quad \bar{F}_t(s) = F_t\big([s, 1]\big).$$

For any profile of sets $(S_t)_{t\in\mathcal{T}} \subset [0, 1]^{|\mathcal{T}|}$ and $T' \subset \mathcal{T}$, let $S_{T'} := (S_t)_{t\in T'}$ and

$$F_{T'}(S_{T'}) := \sum_{t\in T'} F_t(S_t),$$

and, for each $s \in [0, 1]$, let

$$F_{T'}(s) := \sum_{t\in T'} F_t(s) \quad \text{and} \quad \bar{F}_{T'}(s) := \sum_{t\in T'} \bar{F}_t(s).$$

Let $F_\emptyset(\cdot) = \bar{F}_\emptyset(\cdot) = 0$.

Given a measure X of available workers, any choice $X' \sqsubset X$ of the firm must satisfy the following constraint:

$$X'(E) \le \inf_{E'\subset E, E'\in\Sigma} X\big(E \setminus E'\big) + \mathcal{Q}\big(\{t \in T | E' \cap \Theta^t \neq \emptyset\}\big), \quad \forall E \in \Sigma. \tag{S11}$$

LEMMA S3: *Let $F = (F_t)_{t\in\mathcal{T}}$ and $F' = (F'_t)_{t\in\mathcal{T}}$ be the score distributions corresponding to X and $X' \sqsubset X$, respectively. Then, the constraint* (S11) *holds if and only if*

$$F'_{\mathcal{T}}(S_{\mathcal{T}}) \le \Psi_F(S_{\mathcal{T}}) := \min_{T'\subset\mathcal{T}} F_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}) + \mathcal{Q}\big(T'\big), \quad \forall S_{\mathcal{T}} = (S_t)_{t\in\mathcal{T}} \subset [0, 1]^{|\mathcal{T}|}. \tag{S12}$$

PROOF: To prove that (S11) implies (S12), for any $S_{\mathcal{T}} = (S_t)_{t\in\mathcal{T}}$, let $E_t = s^{-1}(S_t) \cap \Theta^t$ for each $t \in \mathcal{T}$. Fix $T' \subset \mathcal{T}$ and set $E = \bigcup_{t\in\mathcal{T}} E_t$ and $E' = \bigcup_{t\in T'} E_t$ in (S11). Then,

$$X'(E) = \sum_{t\in\mathcal{T}} X'(E_t) = \sum_{t\in\mathcal{T}} F'_t(S_t) = F'_{\mathcal{T}}(S_{\mathcal{T}}), \tag{S13}$$

$$X\big(E \setminus E'\big) = \sum_{t\in\mathcal{T}\setminus T'} X(E_t) = \sum_{t\in\mathcal{T}\setminus T'} F_t(S_t) = F_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}). \tag{S14}$$

That (S11) implies (S12) thus follows from observing that $\{t \in \mathcal{T} | E' \cap \Theta^t \neq \emptyset\} \subset T'$.

To prove the converse, if (S11) fails, then there must be E and $E' \subset E$ such that

$$X'(E) > X\big(E \setminus E'\big) + \mathcal{Q}\big(\{t \in \mathcal{T} | E' \cap \Theta^t \neq \emptyset\}\big). \tag{S15}$$

Let $E_t = \Theta^t \cap E$ and $S_t = s(E_t)$ for each $t \in \mathcal{T}$, and $T' = \{t \in \mathcal{T} | E' \cap \Theta^t \neq \emptyset\}$. Then, (S13) easily holds. Also, (S14) holds since $E \setminus E' = E \cap (\bigcup_{t\in\mathcal{T}\setminus T'} \Theta^t) = \bigcup_{t\in\mathcal{T}\setminus T'} (E \cap \Theta^t) = \bigcup_{t\in\mathcal{T}\setminus T'} E_t$. Thus, (S15) means that the inequality (S12) fails. *Q.E.D.*

Given Lemma S3 and the fact that the firm's preference depends only on the score of workers, the firm's optimization problem $[P]$ can be rewritten as

$$[P'] \qquad \max_{(F'_t)_{t\in\mathcal{T}}} \int_0^1 s\, dF'_{\mathcal{T}}(s) \quad \text{subject to (S12).}$$

Once the solution to $[P']$ is obtained, it will be straightforward to find a corresponding solution to the original problem $[P]$, as will be seen later.

Given the definition of $\bar{F}_t$, the set T_k in Greedy Algorithm is a maximal element of

$$\arg \max_{T'\subset\mathcal{T}\setminus(\bigcup_{j=0}^{k-1} T_j)} \inf\left\{ s\in[0,1] \middle| \bar{F}_{T'}(s) < \mathcal{Q}\left(\left(\bigcup_{j=0}^{k-1} T_j\right)\cup T'\right) - \mathcal{Q}\left(\bigcup_{j=0}^{k-1} T_j\right)\right\}, \tag{S16}$$

while s_k is the resulting maximum. (Recall $T_0 = \emptyset$.)

LEMMA S4: *GA yields a unique profile* $(s_k, T_k)_{k=1}^m$.

PROOF: Suppose that there are two profiles given by Greedy Algorithm: $(s_k, T_k)_{k=1}^m$ and $(s'_k, T'_k)_{k=1}^{m'}$. Let $s_0 = s'_0 = 1$ and $T_0 = T'_0 = \emptyset$. Assume wlog that $m \le m'$. For an inductive argument, fix any $k \le m$ and assume that $(s_j, T_j) = (s'_j, T'_j)$, $\forall j < k$. We aim to show that $(s_k, T_k) = (s'_k, T'_k)$. Given the inductive assumption and GA, it is clear that $s_k = s'_k$. Suppose for contradiction that $T_k, T'_k \subset \mathcal{T}\setminus(\bigcup_{j=0}^{k-1} T_j)$ and $T_k \ne T'_k$. By GA, letting $s_k = s'_k = s$ and $\tilde{T} = \bigcup_{j=1}^{k-1} T_k$, we have

$$\sum_{t\in T_k} \bar{F}_t(s) \le \mathcal{Q}(\tilde{T}\cup T_k) - \mathcal{Q}(\tilde{T}) \quad \text{and} \quad \sum_{t\in T'_k} \bar{F}_t(s) \le \mathcal{Q}(\tilde{T}\cup T'_k) - \mathcal{Q}(\tilde{T}) \tag{S17}$$

with equality if $k < m$. Also, we must have

$$\sum_{t\in T_k\cap T'_k} \bar{F}_t(s) \le \mathcal{Q}\big(\tilde{T}\cup(T_k\cap T'_k)\big) - \mathcal{Q}(\tilde{T}). \tag{S18}$$

By definition of T_k and the fact that $T_k \subsetneq T_k \cup T'_k$, we have

$$\begin{aligned}
\sum_{t\in T_k\cup T'_k} \bar{F}_t(s) &< \mathcal{Q}\big(\tilde{T}\cup(T_k\cup T'_k)\big) - \mathcal{Q}(\tilde{T}) \\
&\le \mathcal{Q}(\tilde{T}\cup T_k) - \mathcal{Q}(\tilde{T}) + \mathcal{Q}\big(\tilde{T}\cup T'_k\big) - \mathcal{Q}(\tilde{T}) + \mathcal{Q}(\tilde{T}) - \mathcal{Q}\big(\tilde{T}\cup(T_k\cap T'_k)\big) \\
&\le \sum_{t\in T_k} \bar{F}_t(s) + \sum_{t\in T'_k} \bar{F}_t(s) + \mathcal{Q}(\tilde{T}) - \mathcal{Q}\big(\tilde{T}\cup(T_k\cap T'_k)\big),
\end{aligned}$$

where the weak inequality follows from submodularity of $\mathcal{Q}$ while the equality follows from (S17). Rearranging this equation, we obtain

$$\mathcal{Q}\big(\tilde{T}\cup(T_k\cap T'_k)\big) - \mathcal{Q}(\tilde{T}) < \sum_{t\in T_k\cap T'_k} \bar{F}_t(s),$$

which contradicts (S18).

Last, the inequality $m \leq m'$ must hold as equality, since we have $\bigcup_{k=1}^{m} T'_k = \bigcup_{k=1}^{m} T_k = \mathcal{T}$ by the above induction argument and the definition of m. *Q.E.D.*

Using the profile $(s_k, T_k)_{k=1}^m$ obtained from GA, let us define $F^* = (F_t^*)_{t\in\mathcal{T}}$ as follows: for each $t \in T_k$ and $S \subset [0, 1]$,

$$F_t^*(S) = F_t\big(S \cap [s_k, 1]\big), \tag{S19}$$

that is, the firm hires a worker of ethnic type $t \in T_k$ if and only if her score is above s_k. This score distribution can be generated by the following subpopulation: for any $E \in \Sigma$ and $t \in T_k$,

$$X^*\big(E \cap \Theta^t\big) = X\big(\big\{\theta \in E | \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\big\}\big)$$

and

$$X^*(E) = \sum_{k=1}^{m} \sum_{t\in T_k} X^*\big(E \cap \Theta^t\big). \tag{S20}$$

PROPOSITION S3: *The subpopulation X^* in* (S20) *is a unique solution to* $[P]$.

PROOF: We first prove that $F^* = (F_t^*)_{t\in\mathcal{T}}$ is a solution to $[P']$, which means that X^* is a solution to $[P]$. Afterward, we prove the uniqueness.

We first show that F^* satisfies the feasibility constraint (S12), that is, for any $S_\mathcal{T} = (S_t)_{t\in\mathcal{T}}$,

$$F_T^*(S_T) \leq F_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}) + \mathcal{Q}\big(T'\big), \quad \forall T' \subset \mathcal{T}. \tag{S21}$$

Fix any $T' \subset \mathcal{T}$ and let $T'_k := T' \cap T_k$. for each $k = 1, \dots, m$. Let $s_t = s_k$ for each $t \in T_k$. Note first that

$$F^*_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}) = \sum_{t\in\mathcal{T}\setminus T'} F_t\big(S_t \cap [s_t, 1]\big) \leq \sum_{t\in\mathcal{T}\setminus T'} F_t(S_t) = F_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}). \tag{S22}$$

Next,

$$\begin{aligned} \sum_{t\in T'_k} F_t^*(S_t) &= \sum_{t\in T'_k} F_t\big(S_t \cap [s_k, 1]\big) \\ &\leq \sum_{t\in T'_k} F_t\big([s_k, 1]\big) \leq \mathcal{Q}\left(\left(\bigcup_{i=1}^{k-1} T_i\right) \cup T'_k\right) - \mathcal{Q}\big(\cup_{i=1}^{k-1} T_i\big) \\ &\leq \mathcal{Q}\left(\left(\bigcup_{j=1}^{k-2} T_j\right) \cup T'_{k-1} \cup T'_k\right) - \mathcal{Q}\left(\left(\bigcup_{j=1}^{k-2} T_j\right) \cup T'_{k-1}\right) \\ &\dots \\ &\leq \mathcal{Q}\left(\bigcup_{j=1}^{k} T'_j\right) - \mathcal{Q}\left(\bigcup_{j=1}^{k-1} T'_j\right), \end{aligned} \tag{S23}$$

where the second inequality holds since $T'_k \subset T_k$, while the third to last inequalities hold due to the submodularity. By (S22) and (S23), we get

$$\begin{aligned} F^*_{\mathcal{T}}(S_{\mathcal{T}}) &= \sum_{t\in\mathcal{T}\setminus T'} F^*_t(S_t) + \sum_{k=1}^{m} \sum_{t\in T'_k} F^*_t(S_t) \\ &\leq \sum_{t\in\mathcal{T}\setminus T'} F_t(S_t) + \sum_{k=1}^{m} \left(\mathcal{Q}\left(\bigcup_{j=1}^{k} T'_j \right) - \mathcal{Q}\left(\bigcup_{j=1}^{k-1} T'_j \right) \right) \\ &= \sum_{t\in\mathcal{T}\setminus T'} F_t(S_t) + \mathcal{Q}\left(\bigcup_{j=1}^{m} T'_k \right) \\ &= F_{\mathcal{T}\setminus T'}(S_{\mathcal{T}\setminus T'}) + \mathcal{Q}(T'), \end{aligned}$$

which proves (S21).

To prove the optimality of F^*, note first that (S19) implies

$$\bar{F}^*_{\mathcal{T}}(s) = \begin{cases} \bar{F}_{\mathcal{T}}(s), & \text{if } s \geq s_1, \\ \bar{F}_{\mathcal{T}\setminus(\bigcup_{j=1}^{k-1} T_j)}(s) + \mathcal{Q}\left(\bigcup_{j=1}^{k-1} T_j \right), & \text{if } s \in [s_k, s_{k-1}), k = 2, \ldots, m, \\ \mathcal{Q}(\mathcal{T}), & \text{if } s < s_m, \end{cases} \tag{S24}$$

which in turn implies

$$\bar{F}^*_{\mathcal{T}}(s) = \Psi_F\big([s,1]^{|\mathcal{T}|}\big), \quad \forall s \in [0,1], \tag{S25}$$

that is, the constraint (S12) is binding with $S_t = [s,1]$ for all $t \in \mathcal{T}$ and $s \in [0,1]$. This can be easily seen by setting T' in (S12) as follows: $T' = \emptyset$ if $s \geq s_1$; $T' = \bigcup_{j=1}^{k-1} T_j$ if $s \in [s_k, s_{k-1})$ for some $k \in \{2, \ldots, m\}$; and $T' = \mathcal{T}$ if $s < s_m$. Now, (S25) implies that for any $F' = (F'_t)_{t\in\mathcal{T}}$ satisfying (S12), we have $\bar{F}'_{\mathcal{T}}(s) \leq \bar{F}^*_{\mathcal{T}}(s), \forall s \in [0,1]$. Using this, we obtain

$$\begin{aligned} \int_0^1 s\,dF^*_{\mathcal{T}}(s) &= -s\bar{F}^*_{\mathcal{T}}(s)|_{s=0}^{1} + \int_0^1 \bar{F}^*_{\mathcal{T}}(s)\,ds \\ &= \int_0^1 \bar{F}^*_{\mathcal{T}}(s)\,ds \\ &\geq \int_0^1 \bar{F}'_{\mathcal{T}}(s)\,ds \\ &= \int_0^1 s\,dF'_{\mathcal{T}}(s), \end{aligned} \tag{S26}$$

which means that F^* is a solution to $[P]$.

To prove the uniqueness, let X' be any solution to $[P]$ and $F' = (F'_t)_{t\in\mathcal{T}}$ be the corresponding score distribution, which must thus be a solution to $[P']$. Then, we must have $\bar{F}'_{\mathcal{T}}(s) = \bar{F}^*_{\mathcal{T}}(s) = \Psi_F([s,1]^{|\mathcal{T}|})$ for all $s \in [0,1]$, since otherwise the inequality in (S26) would hold strictly. Next, we prove the following claim:

CLAIM S1: *For all k and $t \in T_k$, $F'_t([s_k, 1]) = F_t([s_k, 1])$ and $F'_t([0, s_k]) = 0$.*

PROOF: Assume that this statement is true up to $k-1$. To show that it also holds for k, observe first that

$$\begin{aligned}\sum_{t \in \mathcal{T} \setminus (\bigcup_{j=1}^{k-1} T_j)} F'_t\big([s_k, 1]\big) &= F'_{\mathcal{T}}\big([s_k, 1]\big) - \sum_{j=1}^{k-1} F'_{T_j}\big([s_k, 1]\big) \\ &= F^*_{\mathcal{T}}\big([s_k, 1]\big) - \sum_{j=1}^{k-1} F_{T_j}\big([s_j, 1]\big) \\ &= \sum_{t \in \mathcal{T} \setminus (\bigcup_{j=1}^{k-1} T_j)} F_t\big([s_k, 1]\big),\end{aligned} \tag{S27}$$

where the second equality holds since $\bar{F}'_{\mathcal{T}} = \bar{F}^*$ and since the induction hypothesis together with the fact that $s_j < s_k, \forall j < k$ implies $F'_{T_j}([s_k, 1]) = F'_{T_j}([s_j, 1]) = F_{T_j}([s_j, 1])$, $\forall j < k$, while the second equality holds since (S16) and (S24) imply

$$\sum_{j=1}^{k-1} \bar{F}_{T_j}(s_j) = \sum_{j=1}^{k-1} \left(\mathcal{Q}\left(\bigcup_{i=0}^{j} T_i \right) - \mathcal{Q}\left(\bigcup_{i=0}^{j-1} T_i \right) \right) = \mathcal{Q}\left(\bigcup_{i=1}^{k-1} T_i \right) = \bar{F}^*_{\mathcal{T}}(s_k) - \bar{F}_{\mathcal{T} \setminus (\bigcup_{i=1}^{k-1} T_j)}(s_k).$$

Since $F'_t([s_k, 1]) \le F_t([s_k, 1]), \forall t$, the equality (S27) implies $F'_t([s_k, 1]) = F_t([s_k.1])$ for all $t \in T_k$. Also, if $F'_t([0, s_k]) > 0$ for some $t \in T_k$, then we have

$$\begin{aligned}\sum_{t \in \bigcup_{j=1}^{k} T_j} F'_t\big([0, 1]\big) &= \sum_{t \in \bigcup_{j=1}^{k} T_j} F'_t\big([0, s_k]\big) + \sum_{t \in \bigcup_{j=1}^{k} T_j} F'_t\big([s_k, 1]\big) \\ &> \sum_{t \in \bigcup_{j=1}^{k} T_j} F'_t\big([s_k, 1]\big) = \sum_{t \in \bigcup_{j=1}^{k} T_j} F_t\big([s_k, 1]\big) = \mathcal{Q}\left(\bigcup_{j=1}^{k} T_j \right),\end{aligned}$$

which contradicts (S12). *Q.E.D.*

For uniqueness, it suffices to prove that for any $E \in \Sigma$ and $t \in \mathcal{T}$, $X'(E \cap \Theta^t) = X^*(E \cap \Theta^t)$. Suppose not for contradiction, and suppose $t \in T_k$. Then, since $F'_t([0, s_k]) = 0$ by Claim S1, we must have

$$X'\big(E \cap \Theta^t\big) < X\big(\{\theta \in E | \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\}\big) = X^*\big(E \cap \Theta^t\big). \tag{S28}$$

Also,

$$X'\big(E^c \cap \Theta^t\big) \le X\big(\{\theta \in E^c | \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\}\big). \tag{S29}$$

Adding up (S28) and (S29) side by side, we obtain

$$F'_t\big([s_k, 1]\big) = X'\big(\Theta^t\big) < X\big(\{\theta \in \Theta | \tau(\theta) = t \text{ and } s(\theta) \in [s_k, 1]\}\big) = F_t\big([s_k, 1]\big),$$

which contradicts Claim S1. *Q.E.D.*

S.6.2. *Proof of Lemma 1*

Consider any subpopulations X and Y with $Y \sqsubset X$ and corresponding score distributions $F = (F_t)_{t\in\mathcal{T}}$ and $G = (G_t)_{t\in\mathcal{T}}$. Note that for any $t \in \mathcal{T}$, Borel set $S \subset [0, 1]$, and $s \in [0, 1]$, we have $F_t(S) \geq G_t(S)$ and $\bar{F}_t(s) \geq \bar{G}_t(s)$. Let $(s_t)_{t\in\mathcal{T}}$ and $(s'_t)_{t\in\mathcal{T}}$ be the cutoff profiles from GA under F and G, respectively.

We first prove substitutability, for which it suffices to show that $s_t \geq s'_t$ for all ethnic types $t \in \mathcal{T}$. To show this, suppose the contrary, that is, there exists an ethnic type $t \in \mathcal{T}$ such that $s_t < s'_t$. Then the set $T^* := \{t \in \mathcal{T} : s_t < s'_t\}$ is nonempty. Fix an ethnic type t^* in this set T^* that has the highest cutoff among those in T^*, that is,

(a) $t^* \in T^*$, and

(b) $s'_{t^*} \geq s'_{t'}$ for every $t' \in T^*$.

Now, let k be the step of GA such that $t^* \in T_k$ under F, and k' be the step of GA such that $t^* \in T_{k'}$ under G, respectively. That is, k and k' are the steps at which some constraint related to type t^* becomes binding under F and G, respectively (or the last step of the algorithm if no constraint related to t^* becomes binding in any step of the algorithm).

Now, note that because t^* satisfies the property in (b) as described above, for every ethnic type t whose constraint is already binding by the beginning of step k' under G, a constraint for that type t is also binding by the beginning of step k under F. More formally, we have $\bar{T}' \subseteq \bar{T}$ for $\bar{T} := \bigcup_{j=1}^{k-1} T_j$ and $\bar{T}' := \bigcup_{j=1}^{k'-1} T'_j$, where T_j and T'_j are the maximal sets that solve the problem given as (S16) in step j of GA under F and G, respectively.[7]

Let T' be the set which is the maximal solution to (S16) at step k' under G. Then, $s^* := s'_{t^*}$ is strictly positive by our maintained assumption $s^* > s_{t^*}$ and the fact $s_{t^*} \geq 0$. Thus, it follows that

$$\bar{G}_{T'}(s^*) = \mathcal{Q}(\bar{T}' \cup T') - \mathcal{Q}(\bar{T}'). \tag{S30}$$

We also note that

$$\bar{G}_{T'\cap\bar{T}}(s^*) \leq \mathcal{Q}(\bar{T}' \cup (T' \cap \bar{T})) - \mathcal{Q}(\bar{T}'), \tag{S31}$$

because T' is a solution of the maximization problem described in (S16), and s^* is the associated time at which a constraint becomes binding in this step. Subtracting (S31) from (S30), we obtain

$$\bar{G}_{T'}(s^*) - \bar{G}_{T'\cap\bar{T}}(s^*) \geq \mathcal{Q}(\bar{T}' \cup T') - \mathcal{Q}(\bar{T}' \cup (T' \cap \bar{T})). \tag{S32}$$

Note that the left-hand side of (S32) satisfies

$$\begin{aligned} \bar{G}_{T'}(s^*) - \bar{G}_{T'\cap\bar{T}}(s^*) &= \bar{G}_{T'\setminus\bar{T}}(s^*) \\ &\leq \bar{F}_{T'\setminus\bar{T}}(s^*), \end{aligned} \tag{S33}$$

where the equality follows from modularity of $\bar{G}$ (with respect to sets of ethnic types) and identity $T' \setminus (T' \cap \bar{T}) = T' \setminus \bar{T}$, while the inequality follows from the assumption that

[7]In case $k = 1$ or $k' = 1$, we take T or T' to be an empty set.

$G \sqsubseteq F$. Note also that the right-hand side of (S32) satisfies

$$\begin{aligned}
&\mathcal{Q}(\bar{T}' \cup T') - \mathcal{Q}(\bar{T}' \cup (T' \cap \bar{T})) \\
&\quad = \mathcal{Q}([\bar{T}' \cup (T' \cap \bar{T})] \cup (T' \setminus \bar{T})) - \mathcal{Q}([\bar{T}' \cup (T' \cap \bar{T})]) \\
&\quad \geq \mathcal{Q}(\bar{T} \cup (T' \setminus \bar{T})) - \mathcal{Q}(\bar{T}),
\end{aligned} \tag{S34}$$

where the equality is an identity and the inequality follows from the fact that $[\bar{T}' \cup (T' \cap \bar{T})] \subseteq \bar{T}$ (which in turn follows from the fact that $\bar{T}'$ is a subset of $\bar{T}$) and submodularity of $\mathcal{Q}$.

Substituting (S33) and (S34) into (S32), we obtain

$$\bar{F}_{T' \setminus \bar{T}}(s^*) \geq \mathcal{Q}(\bar{T} \cup (T' \setminus \bar{T})) - \mathcal{Q}(\bar{T}),$$

which implies $s_t \geq s^* = s'_t$, a contradiction.

To next prove LoAD, consider any subpopulations X and Y with $Y \sqsubset X$ and corresponding score distributions F and G. Let F^* and G^* denote the solution of [P'] under F and G, respectively. The result is then immediate from observing that the total mass hired by the firm is

$$\sum_{t \in \mathcal{T}} F^*_t([0, 1]) = \bar{F}^*_{\mathcal{T}}(0) = \Psi_F([0, 1]^{|T|}) \geq \Psi_G([0, 1]^{|T|}) = \bar{F}^*_{\mathcal{T}}(0) = \sum_{t \in \mathcal{T}} G^*_t([0, 1]),$$

where the inequality follows from the definition of Ψ_F, Ψ_G in (S12) and the fact that $\sum_{t \in \mathcal{T} \setminus T'} F_t([0, 1]) \geq \sum_{t \in \mathcal{T} \setminus T'} G_t([0, 1]), \forall T' \subset \mathcal{T}$.

S.6.3. *Proof of Proposition 2*

To simplify notation, let $M = \underline{M}$, that is, the worker-optimal matching. Fix any individually rational matching $\hat{M}$ such that $\hat{M} \succeq_F M$ and assume that $\bar{F} := \{f' \in F | \hat{M}_{f'} \succ_{f'} M_{f'}\}$ is nonempty. For any f, t, let $M^t_f := M_f(\Theta^t \cap \cdot)$ and $\hat{M}^t_f := \hat{M}_f(\Theta^t \cap \cdot)$. Since G is absolutely continuous, for any f, t, both M^t_f and $\hat{M}^t_f$, being its subpopulations, admit densities, denoted respectively by m^t_f and $\hat{m}^t_f$.

By Proposition S3 in the Supplemental Material, Greedy Algorithm yields a unique optimal choice for each firm. Given this and the fact that $M_f = C_f(M_f)$ and $\hat{M}_f = C_f(M_f \vee \hat{M}_f)$, we may let s^t_f and $\hat{s}^t_f$ denote the cutoffs for each type $t \in T$ for M_f and $\hat{M}_f$ in the sense that $s^t_f = \inf\{s_f(\theta) | \theta \in \Theta^t \text{ and } m^t_f(\theta) > 0\}$ and $\hat{s}^t_f = \inf\{s_f(\theta) | \theta \in \Theta^t \text{ and } \hat{m}^t_f(\theta) > 0\}$.[8]

Because C_f satisfies LoAD by Lemma 1, $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ and $M_f = C_f(M_f)$ imply $M_f(\Theta) \leq \hat{M}_f(\Theta)$ for each $f \in F$. Then, Proposition 2 follows from proving a sequence of claims.

CLAIM S2: *$M_\emptyset = \hat{M}_\emptyset$. Thus, $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$ and $M_f(\Theta) = \hat{M}_f(\Theta), \forall f \in F$.*

[8]These cutoffs are obtained from running Greedy Algorithm with M_f and $M_f \vee \hat{M}_f$ as measures of available workers, respectively. More precisely, we have $s^t_f = s_k$ if $t \in T_k$ in Greedy Algorithm run with M_f as measure of available workers, for instance.

PROOF: Suppose to the contrary that $M_\emptyset \neq \hat{M}_\emptyset$. Then, with their densities denoted by $m_\emptyset$ and $\hat{m}_\emptyset$, $E_\emptyset = \{\theta \in \Theta | m_\emptyset(\theta) > \hat{m}_\emptyset(\theta)\}$ must be a nonempty set of positive (Lebesgue) measure, due to the fact that $M_\emptyset(\Theta) = G(\Theta) - \sum_{f \in F} M_f(\Theta) \geq G(\Theta) - \sum_{f \in F} \hat{M}_f(\Theta) = \hat{M}_\emptyset(\Theta)$. Also, letting $\hat{E}_f = \{\theta \in \Theta | \hat{m}_f(\theta) > m_f(\theta)\}$, there must be at least one firm f for which $E_\emptyset \cap \hat{E}_f$ is a nonempty set of positive measure, since otherwise we would have $\sum_{f' \in \tilde{F}} m_{f'}(\theta) \geq \sum_{f' \in \tilde{F}} \hat{m}_{f'}(\theta)$ for all $\theta \in E_\emptyset$, a contradiction. Now, fixing such a firm f and letting $\tilde{E} = E_\emptyset \cap \hat{E}_f$, define

$$\tilde{m}_f(\theta) = \begin{cases} \min\{m_f(\theta) + m_\emptyset(\theta), \hat{m}_f(\theta)\}, & \text{if } \theta \in \tilde{E}, \\ m_f(\theta), & \text{otherwise,} \end{cases}$$

and let $\tilde{M}_f$ denote the corresponding measure. Note that $\tilde{m}_f(\theta) > m_f(\theta)$ for all $\theta \in \tilde{E}$, and also that $(M_f \vee \tilde{M}_f) = \tilde{M}_f \neq M_f$ and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$. Letting $M'_f = C_f(\tilde{M}_f)$, we show below that f and M'_f are a blocking coalition for M, contradicting the stability of M.

First of all, it follows from revealed preference that $C_f(M_f \vee M'_f) = M'_f$. To show that $M'_f \neq M_f$, note first that $\hat{m}_f(\theta) > m_f(\theta), \forall \theta \in \tilde{E}$ means $(\hat{M}_f \vee M_f)(\tilde{E}) = \hat{M}_f(\tilde{E})$, so

$$R_f(M_f \vee \hat{M}_f)(\tilde{E}) = (M_f \vee \hat{M}_f)(\tilde{E}) - C_f(M_f \vee \hat{M}_f)(\tilde{E}) = \hat{M}_f(\tilde{E}) - \hat{M}_f(\tilde{E}) = 0.$$

Then, since f has a substitutable preference and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$, we have $R_f(\tilde{M}_f)(\tilde{E}) = 0$, which means $M'_f(\tilde{E}) = C_f(\tilde{M}_f)(\tilde{E}) = \tilde{M}_f(\tilde{E}) \neq M_f(\tilde{E})$. It only remains to show that $M'_f \sqsubset D^{\preceq f}(M)$. For this, note that since $\hat{M}$ is individually rational and $\hat{m}_f(\theta) > 0, \forall \theta \in \tilde{E}$, we have $f \succ_\theta \emptyset, \forall \theta \in \tilde{E}$. Given the definition of $\tilde{M}_f$ (i.e., only those added to f are unmatched under M), this implies that $\tilde{M}_f \sqsubset D^{\preceq f}(M)$ and thus $M'_f \sqsubset \tilde{M}_f \sqsubset D^{\preceq f}(M)$. *Q.E.D.*

We then prove the next claim.

CLAIM S3: *For each $f \in \bar{F}$, there must be some t such that $s^t_f < \hat{s}^t_f$.*

PROOF: Suppose to the contrary that $\hat{s}^t_f \leq s^t_f$ for all $t \in T$. Since $\sum_{t \in T} M^t_f(\Theta) = M_f(\Theta) = \hat{M}_f(\Theta) = \sum_{t \in T} \hat{M}^t_f(\Theta)$ and $M_f \neq \hat{M}_f$, there must exist $t \in T$ such that the set $\{\theta \in \Theta^t | s_f(\theta) > s^t_f \geq \hat{s}^t_f \text{ and } m^t_f(\theta) > \hat{m}^t_f(\theta)\}$ has a positive measure. A contradiction then arises since, due to the fact that C_f selects all workers of type t whose scores are above the cutoff $\hat{s}^t_f$ and that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$, the measure of workers of type $\theta \in \Theta^t$ selected when $\hat{M}_f \vee M_f$ is available is equal to $\hat{m}^t_f(\theta) = \max\{\hat{m}^t_f(\theta), m^t_f(\theta)\}$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geq \hat{s}^t_f$, which cannot be smaller than $m^t_f(\theta)$. *Q.E.D.*

CLAIM S4: *For any $f \in \bar{F}$ and $t \in T$, if $\hat{s}^t_f = 0$, then $\hat{M}_f(\Theta^t \cap \cdot) = M_f(\Theta^t \cap \cdot)$.*

PROOF: Let us first observe that for any $f \in \bar{F}$ and t, if $\hat{M}_f(\Theta^t) < M_f(\Theta^t)$, then we have $\hat{s}^t_f > s^t_f$ since, as we argued in the proof of Claim S3, the fact that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ implies that $\hat{m}^t_f(\theta) = \max\{\hat{m}^t_f(\theta), m^t_f(\theta)\} \geq m^t_f(\theta)$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geq \hat{s}^t_f$, so if $\hat{s}^t_f \leq s^t_f$, then we would have a contradiction.

Fix now any $f \in \bar{F}$ and $t \in T$ for which $\hat{s}^t_f = 0$. Since it means $\hat{s}^t_f \leq s^t_f$, we must have $\hat{M}_f(\Theta^t) \geq M_f(\Theta^t)$ according to the above argument. We next show that $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$. Suppose to the contrary that $\hat{M}_f(\Theta^t) > M_f(\Theta^t)$. Then, the fact that $\hat{M}_f(\Theta) = M_f(\Theta)$ by Claim S2 implies that there must exist t' such that $\hat{M}_f(\Theta^{t'}) < M_f(\Theta^{t'})$ and no constraint for t' is binding at $\hat{M}_f$, that is, $\hat{s}^{t'}_f = 0$. To show this, note first that for any $k \in \{1, \dots, m-1\}$,

$$\sum_{t'' \in T_k} \hat{M}_f(\Theta^{t''}) = \mathcal{Q}\left(\bigcup_{j=0}^{k} T_j\right) - \mathcal{Q}\left(\bigcup_{j=0}^{k-1} T_j\right),$$

where m and T_k are as defined in Greedy Algorithm when f chooses $\hat{M}_f$ (given $M_f \vee \hat{M}_f$). Adding up these equalities from $k = 1$ to $m - 1$, we obtain

$$\sum_{t'' \in T^*} \hat{M}_f(\Theta^{t''}) = \mathcal{Q}(T^*), \tag{S35}$$

where $T^* := \bigcup_{k=0}^{m-1} T_k$ represents the set of all ethnic types at least one of whose constraints is binding at $\hat{M}_f$. Also note that, because $\mathcal{Q}$ gives upper-bound constraints for any matching by assumption, we have

$$\sum_{t'' \in T^*} M_f(\Theta^{t''}) \leq \mathcal{Q}(T^*), \tag{S36}$$

so combining (S35) and (S36), we obtain

$$\sum_{t'' \in T^*} \hat{M}_f(\Theta^{t''}) \geq \sum_{t'' \in T^*} M_f(\Theta^{t''}). \tag{S37}$$

Equation (S37) and the assumption that $\hat{M}_f(\Theta^t) > M_f(\Theta^t)$, together with the fact that $M_f(\Theta) = \hat{M}_f(\Theta)$ by Claim S2, imply that

$$\sum_{t'' \in T^{**}} \hat{M}_f(\Theta^{t''}) < \sum_{t'' \in T^{**}} M_f(\Theta^{t''}), \tag{S38}$$

where $T^{**} := T \setminus (T^* \cup \{t\})$ represents the set of ethnic types other than t whose constraints are not binding at $\hat{M}_f$. Equation (S38) implies that there is at least one ethnic type $t' \in T^{**}$ such that

$$\hat{M}_f(\Theta^{t'}) < M_f(\Theta^{t'}), \tag{S39}$$

as desired.

Since $t' \in T^{**}$, that is, t' is unconstrained at $\hat{M}$, all workers of ethnic type t' who are available to f at $\hat{M}$ are hired by f. Furthermore, the firm is faced with a weakly larger measure of workers of ethnic type t' when choosing $\hat{M}$ than at M (recall $\hat{M}_f \succeq_f M_f$). So (S39) cannot hold, a contradiction. Hence, $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$.

Given $\hat{s}^t_f = 0$ (i.e., the lowest possible score), we must have $\max\{\hat{m}^t_f(\theta), m^t_f(\theta)\} = \hat{m}^t_f(\theta)$ for all $\theta \in \Theta^t$. In order that $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$, we must then have $\hat{m}^t_f(\theta) = m^t_f(\theta)$ for (almost) all $\theta \in \Theta^t$. *Q.E.D.*

CLAIM S5: *For any $t \in T$, if there is some $f \in \bar{F}$ such that $\hat{s}^t_f > s^t_f$, then we must have $\hat{s}^t_{f'} > 0, \forall f' \in \bar{F}$.*

PROOF: Fix a firm $f \in \bar{F}$ with $\hat{s}^t_f > s^t_f$. Suppose to the contrary that the set $\bar{F}_0 = \{f' \in \bar{F} | \hat{s}^t_{f'} = 0\}$ is nonempty, and note that $f \notin \bar{F}_0$. Then, let us define $\bar{F}_+ = \bar{F} \backslash \bar{F}_0$ and consider the set

$$\{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (s^t_f, \hat{s}^t_f), \text{ and } s_{f'}(\theta) < \hat{s}^t_{f'} \forall f' \in \bar{F}_+ \setminus \{f\}\}.$$

Since M is stable, all worker types in this set must be matched with f under M, which implies that they cannot be matched with any firm in $\tilde{F} \backslash \bar{F}$ under $\hat{M}$ since $\hat{M}_{f'} = M_{f'}$ for each $f' \in F \backslash \bar{F}$ by assumption and also since $\hat{M}_\emptyset = M_\emptyset$ by Claim S2. Moreover, these workers cannot be matched with any firm $f' \in \bar{F}_+$ under $\hat{M}$ since their scores are below $\hat{s}^t_{f'}$. It thus follows that they must be matched with firms in $\bar{F}_0$ under $\hat{M}$ while being matched with $f \notin \bar{F}_0$ under M, which contradicts Claim S4. *Q.E.D.*

CLAIM S6: *Rich preferences hold.*

PROOF: Fix any $f \in \bar{F}$ and $t \in T$ such that $s^t_f < \hat{s}^t_f$ (given by Claim S3), and let

$$\tilde{\Theta}^t_f := \{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (s^t_f, \hat{s}^t_f), \text{ and } s_{f'}(\theta) < \hat{s}^t_{f'} \ \forall f' \in \bar{F} \setminus \{f\}\}$$

be the set of ethnic type-t workers who prefer f to all other firms and have scores that will make them employable at f under M but not under $\hat{M}$ and not employable at any other firm in $\bar{F}$ under $\hat{M}$. Let $M' := \sum_{t \in T} G(\tilde{\Theta}^t_f \cap \cdot)$ denote the measure of these workers. The full support assumption and the fact (given by Claim S5) that $\hat{s}^t_{f'} > 0, \forall f' \in \bar{F}$ imply that $M'(\Theta) > 0$.

We show that these workers are not employed by any firm in $\bar{F}$ under either $\hat{M}$ or M. It is easy to see that these workers are not employed by any firm in $\bar{F}$ under $\hat{M}$ since their scores are below the cutoffs of these firms at $\hat{M}$. Since $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$, and since $M_f = \hat{M}_f$ for each $f \in F \setminus \bar{F}$, we must have $\sum_{f \in \bar{F}} M_f = \sum_{f \in \bar{F}} \hat{M}_f$. It thus follows that these workers are not employed by firms in $\bar{F}$ under matching M either.

It follows that M' measures the workers who are employed outside $\bar{F}$ under M but available to firm f. Hence, $M' \sqsubset \hat{M}^f_{\bar{F}}$. Since $\hat{s}^t_f > s^t_f$, firm f will wish to replace some of its workers with these workers under M. Hence, $M_f \neq C_f((M_f + \hat{M}^f_{\bar{F}}) \wedge G)$, so the rich preferences property follows. *Q.E.D.*

The above claims complete the proof of the proposition.

S.6.4. *(Counter)Example for Lemma 1: Role of Submodularity*

Suppose that $\mathcal{T} = \{t_1, t_2, t_3\}$ and that $\mathcal{Q}(\{t_1, t_3\}) = \mathcal{Q}(\{t_2, t_3\}) = \mathcal{Q}(\{t_i\}) = 1/2, \forall i$ and $\mathcal{Q}(\mathcal{T}) = \mathcal{Q}(\{t_1, t_2\}) = 1$. It is straightforward to check that this constraint violates the submodularity. Suppose that the subpopulations of available workers are given such that F_{t_i} is uniform on $[0, 1]$ for $i = 1, 3$ while $F_{t_2} = 0$. Clearly, the optimal cutoffs are $s_{t_1} = s_{t_3} = 3/4$ and $s_{t_2} = 0$. Consider next larger subpopulations whose score distributions are uniform on

[0, 1] for all three types. We argue that the optimal cutoffs are $s_{t_1} = s_{t_2} = 1/2$ and $s_{t_3} = 1$, which means that the preference of the firm is not substitutable since the cutoff s_{t_1} decreases from 3/4 to 1/2 as more workers of type t_2 become available. To prove this, let us set up the firm's optimization problem as

$$\max_{(s_{t_i})} \sum_{i=1}^{3} \int_{s_{t_i}}^{1} s\, ds$$

subject to

$$(1 - s_{t_1}) + (1 - s_{t_3}) \leq 1/2, \tag{S40}$$

$$(1 - s_{t_2}) + (1 - s_{t_3}) \leq 1/2. \tag{S41}$$

Note that we ignore all other constraints that can later be verified to be nonbinding. The corresponding Langrangian is

$$\sum_{i=1}^{3} \int_{s_{t_i}}^{1} s\, ds + \lambda_1[s_{t_1} + s_{t_3} - 3/2] + \lambda_2[s_{t_2} + s_{t_3} - 3/2],$$

which yields the first-order conditions given as

$$-s_{t_i} + \lambda_i \geq (=)0 \quad (\text{if } s_{t_i} < 1) \ \text{ for } i = 1, 2,$$
$$-s_{t_3} + \lambda_1 + \lambda_2 \geq (=)0 \quad (\text{if } s_{t_3} < 1).$$

Clearly, $\lambda_1, \lambda_2 > 0$ so (S40) and (S41) must be binding at the optimum. If $s_{t_3} < 1$, then (S40) and (S41) being binding implies $s_{t_1} = s_{t_2} = \frac{3}{2} - s_{t_3} > \frac{1}{2}$ and thus $s_{t_3} = \lambda_1 + \lambda_2 \geq s_{t_1} + s_{t_2} > 1$, a contradiction. So we must have $s_{t_3} = 1$ and thus $s_{t_1} = s_{t_2} = \frac{3}{2} - s_{t_3} = \frac{1}{2}$.

S.7. RESULTS FOR SECTION 7

S.7.1. *Omitted Proofs for Section 7*

PROOF OF LEMMA 5: Let $B(\theta, r) = \{\theta' \in \Theta | d^{\Theta}(\theta', \theta) < r\}$ and $S(\theta, r) = \{\theta' \in \Theta | d^{\Theta}(\theta', \theta) = r\}$ (recall d^{Θ} is a metric for the space Θ). For all $\theta \in \overline{\Theta}_f$ and $r > 0$, there must be some $r_\theta \in (0, r)$ such that $G(S(\theta, r_\theta)) = 0$.[9] This means that $\partial B(\theta, r_\theta) = S(\theta, r_\theta)$ has a zero measure. Consider now a collection $\{B(\theta, r_\theta) | \theta \in \Theta\}$ of open balls that covers $\overline{\Theta}_f$. Since $\overline{\Theta}_f$ is a closed subset of the compact set Θ, it is compact and thus has a finite cover. *Q.E.D.*

PROOF OF LEMMA 6: Consider a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of real numbers converging to 0. Fix any k. Then, by Lemma 5, we can find a finite cover $\{B_\ell^k\}_{\ell=1,\dots,L_k}$ of $\overline{\Theta}_f$ for each k such that, for each ℓ, B_ℓ^k has a radius smaller than ϵ_k and $G(\partial B_\ell^k) = 0$. Define $A_1^k = B_1^k \cap \Theta_f$ and $A_\ell^k = (B_\ell^k \setminus (\bigcup_{\ell'=1}^{\ell-1} B_{\ell'}^k)) \cap \Theta_f$ for each $\ell \geq 2$. Then, $\{A_\ell^k\}_{\ell=1,\dots,L_k}$ constitutes a partition of Θ_f. It is straightforward to see that $G(\partial A_\ell^k) = 0, \forall \ell$, since

[9]To see this, note first that $B(\theta, r) = \bigcup_{\tilde{r}\in[0,r)} S(\theta, \tilde{r})$ and $G(B(\theta, r)) < \infty$. Then, $G(S(\theta, \tilde{r})) > 0$ for at most countably many $\tilde{r}$'s, since otherwise the set $R_n \equiv \{\tilde{r} \in [0, r) | G(S(\theta, \tilde{r})) \geq 1/n\}$ has to be infinite for at least one n, which yields $G(B(\theta, r)) \geq G(\bigcup_{\tilde{r}\in R_n} S(\theta, \tilde{r})) \geq \frac{\infty}{n}$, a contradiction.

$G(\partial B_\ell^k)=0, \forall \ell$, and that $G(\partial \Theta_f)=0$.[10] This implies that $Y(\partial A_\ell^k)=0, \forall \ell$. Given this and the assumption that $Y^q \xrightarrow{w^*} Y$, Condition (e) of Theorem 12 implies that there exists sufficiently large q, denoted q_k, such that, for all $q \geq q_k$,

$$\frac{1}{q} < \frac{\epsilon_k}{L_k} \quad \text{and} \quad \left|Y(A_\ell^k) - Y^q(A_\ell^k)\right| < \frac{\epsilon_k}{L_k}, \quad \forall \ell = 1, \dots, L_k. \tag{S42}$$

Let us choose $(q_k)_{k\in\mathbb{N}}$ to be a sequence that strictly increases with k.

We construct X^q as follows: (i) $X^q(\theta) \leq Y^q(\theta), \forall \theta \in \Theta^q$; (ii) for each $q \in \{q_k, \dots, q_{k+1}-1\}$,

$$X^q(A_\ell^k) = \max\left\{\frac{m}{q} \,\middle|\, m \in \mathbb{N}\cup\{0\} \text{ and } \frac{m}{q} \leq \min\{X(A_\ell^k), Y^q(A_\ell^k)\}\right\}$$
$$\text{for each } \ell = 1, \dots, L_k.$$

It is straightforward to check the existence of X^q that satisfies both (i) and (ii). Note that (i) ensures that $X^q \sqsubset Y^q$ and $X^q(\Theta\backslash\Theta_f) \leq Y^q(\Theta\backslash\Theta_f) = 0$.

We show that for all $q \in \{q_k, \dots, q_{k+1}-1\}$, we have

$$\left|X(A_\ell^k) - X^q(A_\ell^k)\right| < \frac{\epsilon_k}{L_k}. \tag{S43}$$

To see this, consider first the case where $X(A_\ell^k) < Y^q(A_\ell^k)$. Then, by definition of X^q and (S42), we have $0 \leq X(A_\ell^k) - X^q(A_\ell^k) < \frac{1}{q} < \frac{\epsilon_k}{L_k}$. In the case where $X(A_\ell^k) \geq Y^q(A_\ell^k)$, we have $X^q(A_\ell^k) = Y^q(A_\ell^k) \leq X(A_\ell^k) \leq Y(A_\ell^k)$, which implies, by (S42),

$$\left|X(A_\ell^k) - X^q(A_\ell^k)\right| \leq \left|Y(A_\ell^k) - Y^q(A_\ell^k)\right| < \frac{\epsilon_k}{L_k}.$$

Let us now prove that $X^q \xrightarrow{w^*} X$. We do so by invoking (b) of Theorem 12, according to which $X^q \xrightarrow{w^*} X$ if and only if $|\int h\,dX^q - \int h\,dX| \to 0$ as $q \to \infty$, for any uniformly continuous function $h \in C_u(\Theta)$.

Hence, to begin, fix any $h \in C_u(\Theta)$, and fix any $\epsilon > 0$. Next, we define, for each k and $q \in \{q_k, \dots, q_{k+1}-1\}$,

$$\bar{h}_\ell^{q,k} \equiv \frac{\displaystyle\sum_{\theta\in\Theta^q\cap A_\ell^k} X^q(\theta)h(\theta)}{\displaystyle\sum_{\theta\in\Theta^q\cap A_\ell^k} X^q(\theta)} = \frac{\displaystyle\sum_{\theta\in\Theta^q\cap A_\ell^k} X^q(\theta)h(\theta)}{X^q(A_\ell^k)}$$

if $X^q(A_\ell^k) > 0$, and if $X^q(A_\ell^k) = 0$, then define $\bar{h}_\ell^{q,k} \equiv h(\theta)$ for some arbitrarily chosen $\theta \in A_\ell^k$.

[10]The latter fact holds since $\Theta_f = \bigcup_{P\in\mathcal{P}:f\succ\emptyset}\Theta_P$ and thus $\partial\Theta_f \subset \bigcup_{P\in\mathcal{P}:f\succ_P\emptyset}\partial\Theta_P$, which implies

$$G(\partial\Theta_f) \leq G\Big(\bigcup_{P\in\mathcal{P}:f\succ_P\emptyset}\partial\Theta_P\Big) \leq \sum_{P\in\mathcal{P}:f\succ_P\emptyset} G(\partial\Theta_P) = 0.$$

Note that $C_u(\Theta)$ is endowed with the sup norm $\|\cdot\|_\infty$ and $\|h\|_\infty$ is finite for any $h \in C_u(\Theta)$. Thus, there exists sufficiently large $K \in \mathbb{N}$ that, for all $k > K$ and $q \in \{q_k, \dots, q_{k+1} - 1\}$,

$$\|h\|_\infty \epsilon_k < \frac{\epsilon}{2} \quad \text{and} \quad \sum_{\ell=1}^{L_k} \Big(\sup_{\theta \in A_\ell^k} \big| \bar{h}_\ell^{q,k} - h(\theta) \big| \Big) X\big(A_\ell^k\big) < \frac{\epsilon}{2}, \tag{S44}$$

where the latter inequality is possible since the expression in the parentheses can be made arbitrarily small by choosing sufficiently large k due to the uniform continuity of h and the fact that $A_\ell^k \subset B_\ell^k$ while B_ℓ^k has a radius smaller than ϵ_k with $\epsilon_k \to 0$ as $k \to \infty$.

Then, for any $q > Q := q_K$, there exists $k > K$ satisfying $q \in \{q_k, \dots, q_{k+1} - 1\}$ such that

$$\begin{aligned}
&\left| \int h\,dX^q - \int h\,dX \right| \\
&\quad = \left| \int_{\theta \in \Theta_f} h\,dX^q - \int_{\theta \in \Theta_f} h\,dX \right| \\
&\quad = \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X^q\big(A_\ell^k\big) - \int_{\theta \in \Theta_f} h\,dX \right| \\
&\quad \le \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} \big(X^q\big(A_\ell^k\big) - X\big(A_\ell^k\big) \big) \right| + \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X\big(A_\ell^k\big) - \int_{\theta \in \Theta_f} h\,dX \right| \\
&\quad \le \sum_{\ell=1}^{L_k} \|h\|_\infty \big| X^q\big(A_\ell^k\big) - X\big(A_\ell^k\big) \big| + \left| \sum_{\ell=1}^{L_k} \int_{\theta \in \Theta_f} \bar{h}_\ell^{q,k} \mathbb{1}_{A_\ell^k}\,dX - \sum_{\ell=1}^{L_k} \int_{\theta \in \Theta_f} h \mathbb{1}_{A_\ell^k}\,dX \right| \\
&\quad \le \|h\|_\infty \epsilon_k + \sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} \big| \bar{h}_\ell^{q,k} - h(\theta) \big| X\big(A_\ell^k\big) \\
&\quad \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where the first equality holds since $X(\Theta \backslash \Theta_f) = X^q(\Theta \backslash \Theta_f) = 0$, while the third and fourth inequalities follow from (S43) and (S44), respectively. *Q.E.D.*

PROOF OF LEMMA 7: Letting $Z^q = Y^q - X^q$ and $Z = Y - X$, we have $Z^q \xrightarrow{w^*} Z$ because of the fact that for any $h \in C_u(\Theta)$,

$$\int_\Theta h\,dZ^q = \int_\Theta h\,dY^q - \int_\Theta h\,dX^q \to \int_\Theta h\,dY - \int_\Theta h\,dX = \int_\Theta h\,dZ$$

and (b) of Theorem 12. Since $Z^q = Y^q - X^q \in \mathcal{X}$, $Z^q \xrightarrow{w^*} Z$, and $\mathcal{X}$ is compact, we have $Z \in \mathcal{X}$, which implies that $Z(E) = Y(E) - X(E) \ge 0$ for all $E \in \Sigma$, as desired. *Q.E.D.*

S.7.2. *Proof for the Existence of ϵ-Distance Stable Matching*

Let us reiterate the definition of ϵ-distance stability:[11] A matching $M^q \in (\mathcal{X}^q)^{n+1}$ in economy Γ^q is **ϵ-distance stable** if (i) for each $f \in F$, $M_f^q \in C_f^q(M_f^q)$; (ii) for each $P \in \mathcal{P}$, $M_f^q(\Theta_P) = 0, \forall f \prec_P \emptyset$; and (iii′) $d(\tilde{M}_f^q, M_f^q) < \epsilon$ for any coalition f and $\tilde{M}_f^q \in \mathcal{X}^q$ that blocks M^q in the sense that $\tilde{M}_f^q \sqsubset D^{\preceq f}(M^q)$ and $u_f(\tilde{M}_f^q) > u_f(M_f^q)$.

PROPOSITION S4: *Suppose that there exists a stable matching in* Γ *such that* $C_f(M_f) = \{M_f\}, \forall f \in F$. *Then, for any* $\epsilon > 0$, *there is* $Q \in \mathbb{N}$ *such that for all* $q > Q$, *there exists an* ϵ*-distance stable matching.*

This result follows directly from combining the following two lemmas.[12]

LEMMA S5: *Consider any stable matching* M *in* Γ *such that* $C_f(M_f) = \{M_f\}, \forall f \in F$. *Then, there exists a sequence* $(M^q)_{q\in\mathbb{N}}$ *such that* $M^q \xrightarrow{w^*} M$, *while* $M^q = (M_f^q)_{f\in\tilde{F}}$ *is a feasible and individually rational matching in* Γ^q.

PROOF: Given M, let us construct the matchings $\tilde{M}^q$ and M^q as in the proof of Lemma 8. It suffices to show that M_f^q converges to M_f since M^q is feasible and individually rational in Γ^q. To do so, we use the following fact: If every subsequence of sequence $(M_f^q)_{q\in\mathbb{N}}$ has a further subsequence that converges to M_f, then M_f^q converges to M_f. Consider any subsequence $(M_f^{k_m})_{m\in\mathbb{N}}$, which must then have a further subsequence, denoted $(M_f^{\ell_m})_{m\in\mathbb{N}}$, converging to some $\hat{M}_f$ since the sequence $(M_f^{k_m})_{m\in\mathbb{N}}$ lies in the compact space $\mathcal{X}$. Suppose for a contradiction that $\hat{M}_f \neq M_f$. Note first that $M_f^{\ell_m} \sqsubset \tilde{M}_f^{\ell_m}, \forall m \in \mathbb{N}$ (since $M_f^q \in C_f^q(\tilde{M}_f^q), \forall q \in \mathbb{N}$) and $\tilde{M}_f^{\ell_m} \xrightarrow{w^*} M_f$, which implies by Lemma 7 that $\hat{M}_f \sqsubset M_f$. Thus, we must have $u_f(\hat{M}_f) = u_f(M_f) - \epsilon$ for some $\epsilon > 0$ since $\hat{M}_f \neq C_f(M_f) = M_f$. By Lemma 8, we can find $Q \in \mathbb{N}$ such that for all $q > Q$,

$$u_f(M_f) < u_f\big(M_f^q\big) + \frac{\epsilon}{2}. \tag{S45}$$

Also, since $M_f^{\ell_m} \xrightarrow{w^*} \hat{M}_f$, we can find a sufficiently large $\ell_m > Q$ such that $u_f(M_f^{\ell_m}) < u_f(\hat{M}_f) + \frac{\epsilon}{2} = u_f(M_f) - \frac{\epsilon}{2}$, which contradicts (S45). *Q.E.D.*

LEMMA S6: *Consider the sequence* $(M^q)_{q\in\mathbb{N}}$ *in Lemma* S5. *For any* $\epsilon > 0$, *there is* $Q \in \mathbb{N}$ *such that for all* $q > Q$, M^q *is an* ϵ*-distance stable matching.*

PROOF: Let $\mathcal{B}_f^q$ denote the set of all blocking coalitions involving f under M^q: that is, $\mathcal{B}_f^q = \{\tilde{M} \in \mathcal{X}^q | \tilde{M} \sqsubset D^{\preceq f}(M_f^q)$ and $u_f(\tilde{M}) > u_f(M^q)\}$. Since $\mathcal{B}_f^q$ is finite for each q, the set $\mathcal{B}_f := \bigcup_{q\in\mathbb{N}} \mathcal{B}_f^q$ is countable. One can index the blocking coalitions in $\mathcal{B}_f$ to form a sequence $(\tilde{M}^k)_{k\in\mathbb{N}}$ such that, for any $\tilde{M}^k \in \mathcal{B}_f^q$ and $\tilde{M}^{k'} \in \mathcal{B}_f^{q'}$ with $q < q'$, we have $k' > k$. Define $q(k)$ to be such that $\tilde{M}^k \in \mathcal{B}_f^{q(k)}$. We show that $\tilde{M}^k \xrightarrow{w^*} M_f$. If not,

[11] The definition of ϵ-distance stability is introduced in footnote 57 of the main paper.
[12] These lemmas are also used to prove Theorem 9.

there must be a subsequence $(\tilde{M}^{k_m})_{m\in\mathbb{N}}$ that converges to some $M'\in\mathcal{X}$ with $M'\neq M_f$. To draw a contradiction, note first that since $D^{\preceq f}(\cdot)$ is continuous and $M^q\xrightarrow{w^*}M$, we have $D^{\preceq f}(M^q)\xrightarrow{w^*}D^{\preceq f}(M)$. Combining this with the fact that $\tilde{M}^{k_m}\xrightarrow{w^*}M'$ and $\tilde{M}^{k_m}\sqsubset D^{\preceq f}(M^{q(k_m)})$, and invoking Lemma 7, we obtain $M'\sqsubset D^{\preceq f}(M)$, which implies that $u_f(M_f)-\epsilon'>u_f(M')+\epsilon'$ for some $\epsilon'>0$, since C_f chooses a uniquely utility-maximizing subpopulation. Since $\tilde{M}^{k_m}\xrightarrow{w^*}M'$ and $M^q\xrightarrow{w^*}M_f$, we can find sufficiently large m such that $u_f(M_f^{q(k_m)})>u_f(M_f)-\epsilon'>u_f(M')+\epsilon'>u_f(\tilde{M}^{q(k_m)})$, which contradicts the fact that $\tilde{M}^{q(k_m)}\in\mathcal{B}_f^{q(k_m)}$. This establishes that $\tilde{M}^k\xrightarrow{w^*}M_f$. Using this and the fact that $M_f^q\xrightarrow{w^*}M_f$, one can choose sufficiently large K such that, for all $k>K$, we have $d(\tilde{M}^k,M_f)<\frac{\epsilon}{2}$ and $d(M_f,M_f^{q(k)})<\frac{\epsilon}{2}$, which implies that $d(\tilde{M}^k,M_f^{q(k)})<d(\tilde{M}^k,M_f)+d(M_f,M_f^{q(k)})<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$. This means that for all $q>q(K)$ and $\tilde{M}\in\mathcal{B}_f^q$, we have $d(\tilde{M},M_f^q)<\epsilon$, showing that M^q is an ϵ-distance stable matching. *Q.E.D.*

S.7.3. *(Counter)Example for Theorem 9*

In this section, we provide an example that shows the assumption, $C_f(\underline{M}_f)=\{\underline{M}_f\},\forall f\in\tilde{F}$, is necessary for Part 2 of Theorem 9.

Assume that $\Theta=\{\theta_1,\theta_2,\theta_3\}$ and that $G^q(\theta_1)=G^q(\theta_2)=\frac{n_q}{q}$ and $G^q(\theta_3)=\frac{q-2n_q}{q}$, where n_q is a positive integer satisfying $\frac{n_q}{q}<\frac{1}{3}$ and $\lim_{q\to\infty}\frac{n_q}{q}=\frac{1}{3}$, which implies $G(\theta_i)=\frac{1}{3},\forall i$. Assume also that in any finite economy Γ^q and limit economy Γ, there is a single firm f which is acceptable to all three types of workers and whose utility function is given as

$$u_f(x_1,x_2,x_3)=\max\{x_1,x_2\}-x_1x_2\left(\frac{1}{3}-x_1\right)\left(\frac{1}{3}-x_2\right)+x_3, \tag{S46}$$

where x_i is the measure of type θ_i. Given this, we have $\underline{M}_f(\theta_i)=\frac{1}{3},\forall i$ while $C_f(\underline{M}_f)=\{(x_1,x_2,x_3)\mid\max\{x_1,x_2\}=x_3=\frac{1}{3}\text{ and }x_1,x_2\geq 0\}$ so the assumption $C_f(\underline{M}_f)=\{\underline{M}_f\}$ fails. In the finite economy Γ^q, the δ-stability requires that either

$$G^q(\theta_1)+G^q(\theta_3)-\delta\leq M_f^q(\theta_1)+M_f^q(\theta_3)\leq G^q(\theta_1)+G^q(\theta_3)\quad\text{and}\quad M_f^q(\theta_2)=0 \tag{S47}$$

or

$$G^q(\theta_2)+G^q(\theta_3)-\delta\leq M_f^q(\theta_2)+M_f^q(\theta_3)\leq G^q(\theta_2)+G^q(\theta_3)\quad\text{and}\quad M_f^q(\theta_1)=0, \tag{S48}$$

while $M_f^q(\theta_i)\geq 0,\forall i$. To see this, note that if both $M_f^q(\theta_2)$ and $M_f^q(\theta_1)$ were positive, then the firm could drop the entire mass of either type-θ_1 or type-θ_2 workers to (strictly) increase the second term in (S46) without affecting any other terms. If, for instance, $M_f^q(\theta_1)=0$, then the firm's utility becomes $M_f^q(\theta_1)+M_f^q(\theta_3)$, so the δ-stability requires (S47). Observe now that for any δ-stable matching M^q satisfying (S47), there is another δ-stable matching $\tilde{M}^q$ satisfying (S48) such that $M_f^q(\theta_1)=\tilde{M}_f^q(\theta_2)$ and $M_f^q(\theta_3)=\tilde{M}_f^q(\theta_3)$. However, for small ϵ, neither matching is ϵ-worker optimal stable in Γ^q since the interests of types θ_1 and θ_2 are sharply opposed across the two matchings.

S.8. ANALYSIS FOR SECTION 8.1

S.8.1. *Proofs*

PROOF OF THEOREM 2: To prove (i), suppose a matching M is stable and population-proportional. We shall show that M satisfies the property (ii) of Definition 11. The population proportionality of M, equivalently equality (13), implies that, if $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$ for any $\theta, \theta' \in \Theta_f^k$, then we must have $M_f(\theta) = D^{\preceq f}(M)(\theta)$, or else $\frac{M_f(\theta)}{G(\theta)} = \alpha_f^k$, but in that case, we have a contradiction since $\alpha_f^k \geq \frac{M_f(\theta')}{G(\theta')}$. Then, by definition of $D^{\preceq f}$,

$$M_f(\theta) = D^{\preceq f}(M)(\theta) = \sum_{f' \in \tilde{F}: f' \preceq f} M_{f'}(\theta) = M_f(\theta) + \sum_{f' \in \tilde{F}: f' \prec f} M_{f'}(\theta),$$

so $\sum_{f' \in \tilde{F}: f' \prec f} M_{f'}(\theta) = 0$. We have thus proven that M is strongly stable.

To prove (ii), fix any mechanism φ that implements a strongly stable matching for any measure. Suppose for contradiction that inequality (12) fails for some measure $G \in \overline{\mathcal{X}}$, for some a, P, P', with (a, P) and (a, P') in the support of G, and for some f. Then, let f be the most preferred firm (or the outside option) at P among those for which inequality (12) fails. Then,

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} < \sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')}, \tag{S49}$$

while

$$\sum_{f': f' \succeq_P f_-^P} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} \geq \sum_{f': f' \succeq_P f_-^P} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')},$$

so it follows that

$$\frac{\varphi_f(G)(a, P)}{G(a, P)} < \frac{\varphi_f(G)(a, P')}{G(a, P')}. \tag{S50}$$

By the strong stability of $\varphi(G)$ and the fact that (a, P) and (a, P') are in the same indifference class for firm f by assumption, inequality (S50) holds only if $\sum_{f': f \succ_P f'} \varphi_{f'}(G)(a, P) = 0$. Thus, because $\sum_{f' \in \tilde{F}} \frac{\varphi_{f'}(G)(a,P)}{G(a,P)} = 1$ as $\varphi(G)$ is a matching, we obtain

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} = 1.$$

This equality contradicts inequality (S49) because the right-hand side of inequality (S49) cannot be strictly larger than 1 as $\varphi(G)$ is a matching, which completes the proof. *Q.E.D.*

Proof of Theorem 11 requires several lemmas.

LEMMA S7: *The correspondence defined in* (11) *is convex-valued and upper hemicontinuous, and satisfies the revealed preference property.*

PROOF: To first show that C_f is convex-valued, for any given X, consider any $X', X'' \in C_f(X)$. Note first that $X', X'' \sqsubseteq X$ implies $\lambda X' + (1-\lambda)X'' \sqsubseteq X$. Also, for any $\lambda \in [0,1]$ and $k \in I_f$,

$$\sum_{\theta\in\Theta_f^k}\bigl(\lambda X' + (1-\lambda)X''\bigr)(\theta) = \lambda\sum_{\theta\in\Theta_f^k} X'(\theta) + (1-\lambda)\sum_{\theta\in\Theta_f^k} X''(\theta) = \Lambda_f^k(X),$$

where the second equality holds since the assumption that $X', X'' \in C_f(X)$ implies $\Lambda_f^k(X) = \sum_{\theta\in\Theta_f^k} X'(\theta) = \sum_{\theta\in\Theta_f^k} X''(\theta)$. Thus, $\lambda X' + (1-\lambda)X'' \in C_f(X)$.

To next show the upper hemicontinuity, consider two sequences $(X^\ell)_{\ell\in\mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell\in\mathbb{N}}$ converging to some X and $\tilde{X}$, respectively, such that for each ℓ, $\tilde{X}^\ell \in C_f(X^\ell)$, that is, $\tilde{X}^\ell \sqsubseteq X^\ell$ and $\Lambda_f^k(X^\ell) = \sum_{\theta\in\Theta_f^k} \tilde{X}^\ell(\theta), \forall k \in I_f$. Since Λ_f is continuous, we have $\Lambda_f^k(X) = \lim_{\ell\to\infty} \Lambda_f^k(X^\ell) = \lim_{\ell\to\infty} \sum_{\theta\in\Theta_f^k} \tilde{X}^\ell(\theta) = \sum_{\theta\in\Theta_f^k} \tilde{X}(\theta)$, which, together with the fact that $\tilde{X} \sqsubseteq X$, means that $\tilde{X} \in C_f(X)$, establishing the upper hemicontinuity of C_f.[13] To show the revealed preference property, let $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, and suppose $C_f(X) \cap \mathcal{X}_{X'} \neq \emptyset$. Consider any $Y \in C_f(X)$ such that $Y(\theta) \leq X'(\theta)$ for all θ. Then, $\Lambda_f^k(X) = \sum_{\theta\in\Theta_f^k} Y(\theta) \leq \sum_{\theta\in\Theta_f^k} X'(\theta)$ for all $k \in I_f$. By the revealed preference property of Λ_f, it follows that $\Lambda_f(X') = \Lambda_f(X)$. Therefore, Y satisfies $\sum_{\theta\in\Theta_f^k} Y(\theta) = \Lambda_f^k(X) = \Lambda_f^k(X')$ for all $k \in I_f$, which implies that $Y \in C_f(X')$ and thus $C_f(X) \cap \mathcal{X}_{X'} \subseteq C_f(X')$. To show $C_f(X) \cap \mathcal{X}_{X'} \supseteq C_f(X')$, consider any $Y \in C_f(X')$ and $\tilde{X} \in \mathcal{X}_{X'}$ such that $\tilde{X} \in C_f(X)$. By the previous argument, we have $\tilde{X} \in C_f(X')$, which implies that for each $f \in F$ and $k \in I_f$, $\sum_{\theta\in\Theta_f^k} Y(\theta) = \sum_{\theta\in\Theta_f^k} \tilde{X}(\theta)$. Since $\tilde{X} \in C_f(X)$, this means that $Y \in C_f(X)$ and thus $Y \in C_f(X) \cap \mathcal{X}_{X'}$. Therefore, we conclude that $C_f(X') = C_f(X) \cap \mathcal{X}_{X'}$ as desired. *Q.E.D.*

From now, we establish a couple of lemmas (Lemmas S8 and S9) and use them to prove Theorem 11. To do so, define a correspondence B_f from $\mathcal{X}$ to itself as follows:

$$B_f(X) := \bigl\{X' \sqsubset X \mid \text{for each } k \in I_f, \text{ there is some } \alpha^k \in [0,1] \text{ such that } X'(\theta) = \min\bigl\{X(\theta), \alpha^k G(\theta)\bigr\} \text{ for all } \theta \in \Theta_f^k\bigr\}. \tag{S51}$$

We then modify the choice correspondence C_f in (11) to

$$\tilde{C}_f(X) = C_f(X) \cap B_f(X), \tag{S52}$$

for every $f \in F$ while we let $\tilde{C}_\emptyset = C_\emptyset$.

LEMMA S8: *For any $X \sqsubset G$, $\tilde{C}_f(X)$ is nonempty and a singleton set (i.e., $\tilde{C}_f$ is a function). Also, $\tilde{C}_f$ satisfies the revealed preference property.*

[13] The argument for $\tilde{X} \sqsubseteq X$ is that for each $\theta \in \Theta$, $\tilde{X}^\ell(\theta) \leq X^\ell(\theta)$, so taking the limit with respect to ℓ yields $\tilde{X}(\theta) \leq X(\theta)$.

PROOF: We first establish that for X, $\tilde{C}_f(X)$ is a singleton set. To do so, for any $X \in \mathcal{X}$, $f \in F$, $k \in I_f$, and $\alpha^k \in [0,1]$, define $\zeta_f^k(\alpha^k) := \sum_{\theta \in \Theta_f^k} \min\{X(\theta), \alpha^k G(\theta)\}$. From now on, we assume $C_f(X) \neq \{X\}$ since, if $C_f(X) = \{X\}$, then we have $\tilde{C}_f(X) = \{X\}$, a singleton set as desired. We show that there exists a unique $\hat{\alpha}^k$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Lambda_f^k(X)$, which means that $\tilde{C}_f(X)$ is a singleton set. First, we must have $\hat{\alpha}^k < \max_{\theta \in \Theta_f^k} X(\theta)$ since otherwise $\zeta_f^k(\hat{\alpha}^k) = \sum_{\theta \in \Theta_f^k} X(\theta) > \Lambda_f^k(X)$ (which follows from the assumption that $C_f(z) \neq \{X\}$ and thus, for any $X' \in C_f(X)$, $X' \sqsubset X$ and $X' \neq X$). Next, observe that $\zeta_f^k(\cdot)$ is strictly increasing in the range $[0, \max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)})$. Then, the continuity of ζ_f^k, along with the fact that $\zeta_f^k(0) = 0$ and $\zeta_f^k(\max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)}) > \Lambda_f^k(X)$, implies that there is a unique $\hat{\alpha}^k \in [0, \max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)})$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Lambda_f^k(X)$.

To show the revealed preference property, consider any $X, X', X'' \in \mathcal{X}$ such that $\tilde{C}_f(X) = \{X'\}$ and $X' \sqsubset X'' \sqsubset X$. Since we already know that $C_f(\cdot)$ satisfies the revealed preference property, we have $X' \in C_f(X'')$. It suffices to show that $X' \in B_f(X'')$, since it means $\tilde{C}_f(X'') = \{X'\}$, from which the revealed preference property follows. To do so, note that $X' \in B_f(X)$ means that $X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}$ for each k and $\theta \in \Theta_f^k$. Then, since $X(\theta) \geq X''(\theta) \geq X'(\theta)$ and $\alpha^k G(\theta) \geq X'(\theta)$, we have

$$X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\} \geq \min\{X''(\theta), \alpha^k G(\theta)\} \geq X'(\theta),$$

so $X'(\theta) = \min\{X''(\theta), \alpha^k G(\theta)\}$ as desired. *Q.E.D.*

LEMMA S9: *Any stable matching in the economy* $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ *is stable and population-proportional in the economy* $(G, F, \mathcal{P}_\Theta, C_F)$.[14]

PROOF: Consider a stable matching $M = (M_f)_{f \in \tilde{F}}$ in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ and let $X_f = D^{\preceq f}(M)$ for each $f \in \tilde{F}$. We first show that M is stable in $(G, F, \mathcal{P}_\Theta, C_F)$. It is straightforward, thus omitted, to check the individual rationality. To check the condition of no blocking coalition, suppose to the contrary that there is a blocking pair f and M'_f, which means that $M'_f \sqsubset X_f$, $M'_f \in C_f(M'_f \vee M_f)$, and $M_f \notin C_f(M'_f \vee M_f)$. Given this, by Lemma S8, there exists $\tilde{M}_f$ such that $\tilde{C}_f(M'_f \vee M_f) = \{\tilde{M}_f\}$. First, by the revealed preference property of $\tilde{C}_f$ and the fact that $\tilde{M}_f \sqsubset (\tilde{M}_f \vee M_f) \sqsubset (M'_f \vee M_f)$, we have $\tilde{M}_f \in \tilde{C}_f(\tilde{M}_f \vee M_f)$ and $M_f \notin \tilde{C}_f(\tilde{M}_f \vee M_f)$. Second, since $M_f \sqsubset X_f$ and $M'_f \sqsubset X_f$, we have $\tilde{M}_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$. In sum, f and $\tilde{M}_f$ form a blocking pair in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, which is a contradiction.

To show the population proportionality of M, observe that since M is stable in the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, we have $M_f = \tilde{C}_f(D^{\preceq f}(M)) = C_f(D^{\preceq f}(M)) \cap B_f(D^{\preceq f}(M))$ for each $f \in F$. Thus, $M_f \in B_f(D^{\preceq f}(M))$, that is, there is some α^k for each $k \in I_f$ such that (13) holds. *Q.E.D.*

PROOF OF THEOREM 11: First, consider the case in which each firm's preference satisfies continuity. Given Lemma S9, it suffices to establish the existence of stable matching in

[14]The economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ is a hypothetical economy that is identical to the original economy, except that the firms' choice correspondences C_F are replaced by $\tilde{C}_F$, which is defined in (S52).

the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$. For doing so, we prove the continuity of $\tilde{C}_f$ and invoke Theorem 2. The continuity of $\tilde{C}_f = C_f \cap B_f$ follows if both C_f and B_f are shown to be upper hemicontinuous, since the intersection of a family of closed-valued upper hemicontinuous correspondences, one of which is also compact-valued, is upper hemicontinuous (see 16.25 Theorem of Aliprantis and Border (2006), for instance), implying that $\tilde{C}_f$, which is a singleton-valued correspondence by Lemma S8, is continuous.

Since C_f is upper hemicontinuous by Lemma S7, it remains to show that B_f is upper hemicontinuous. Consider sequences $(X^\ell)_{\ell\in\mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell\in\mathbb{N}}$ with $\tilde{X}^\ell \in B_f(X^\ell), \forall \ell$, converging weakly to X and $\tilde{X}$, respectively. So, for each $k \in I_f$, there is a sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ such that $\tilde{X}^\ell(\theta) = \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}, \forall \theta \in \Theta^k_f$. For each k, let α^k be a limit to which a subsequence of the sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ converges. We claim that $\tilde{X}(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}, \forall \theta \in \Theta^k_f$. If $\tilde{X}(\theta) > \min\{X(\theta), \alpha^k G(\theta)\}$, then one can find sufficiently large ℓ to make $\tilde{X}^\ell(\theta)$, $X^\ell(\theta)$, and α^k_ℓ close to $\tilde{X}(\theta)$, $X(\theta)$, and α^k, respectively, so that $\tilde{X}^\ell(\theta) > \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}$, which is a contradiction. The same argument applies to the case where $\tilde{X}(\theta) < \min\{X(\theta), \alpha^k G(\theta)\}$.

Second, consider the case in which each firm's preference satisfies substitutability. Let $\tilde{C}_f$ be the augmented choice of f and $\tilde{R}_f$ the corresponding augmented rejection function. For each $f \in F$ and $k \in I_f$, let $\rho^k_f : \mathcal{X} \to \mathbb{R}_+$ denote firm f's rejection of total measure of workers in the indifference class Θ^k_f. Formally, define $\rho^k_f(X) := \sum_{\theta\in\Theta^k_f} X(\theta) - \Lambda^k_f(X)$ for each X.

Without loss of generality, fix $k \in I_f$ and consider X, X' with $X \sqsubseteq X'$ and $X \neq X'$ such that $X(\theta) = X'(\theta)$ for every $\theta \notin I^k_f$. First, consider $k' \neq k$. Then, by substitutability of Λ, we have $\rho^{k'}_f(X) \le \rho^{k'}_f(X')$. Because $\sum_{\theta\in\Theta^{k'}_f} X(\theta) = \sum_{\theta\in\Theta^{k'}_f} X'(\theta)$ by assumption, it follows that

$$\Lambda^{k'}_f(X) = \sum_{\theta\in\Theta^{k'}_f} X(\theta) - \rho^{k'}_f(X) \ge \sum_{\theta\in\Theta^{k'}_f} X'(\theta) - \rho^{k'}_f\bigl(X'\bigr) = \Lambda^{k'}_f\bigl(X'\bigr).$$

Hence, $\alpha^{k'}_f \in [0, 1]$ such that

$$\Lambda^{k'}_f(X) = \sum_{\theta\in\Theta^{k'}_f} \min\bigl\{X(\theta), \alpha^k_f G(\theta)\bigr\},$$

and $\bar{\alpha}^{k'}_f \in [0, 1]$ such that

$$\Lambda^{k'}_f\bigl(X'\bigr) = \sum_{\theta\in\Theta^{k'}_f} \min\bigl\{X'(\theta), \bar{\alpha}^k_f G(\theta)\bigr\},$$

have a relationship $\alpha^{k'}_f \ge \bar{\alpha}^{k'}_f$ (to see this, recall $X(\theta) = X'(\theta)$ for any $\theta \in \Theta^{k'}_f$ by assumption and note that the right-hand sides of these equations are nondecreasing in $\alpha^{k'}_f$ and $\bar{\alpha}^{k'}_f$, respectively). This implies $\tilde{R}_f(X)(\theta) \le \tilde{R}_f(X')(\theta)$ for all $\theta \in \Theta^{k'}_f$, as desired.

Second, consider k and investigate the following cases:

1. Suppose $\Lambda^k_f(X) = \sum_{\theta\in\Theta^k_f} X(\theta)$. Then, clearly, $\rho^k_f(X) = \sum_{\theta\in\Theta^k_f} X(\theta) - \Lambda^k_f(X) = 0$, and thus $\tilde{R}_f(X)(\theta) = 0 \le \tilde{R}_f(X')(\theta)$ for all $\theta \in \Theta^k_f$, as desired.

2. Suppose $\Lambda_f^k(X) < \sum_{\theta\in\Theta_f^k} X(\theta)$. Then, the following holds:

CLAIM S7: $\Lambda_f^k(X) = \Lambda_f^k(X')$.

PROOF: Suppose for contradiction that $\Lambda_f^k(X) \neq \Lambda_f^k(X')$. First, we cannot have $\Lambda_f^k(X') \in [0, \sum_{\theta\in\Theta_f^k} X(\theta)]$, since it would imply $\Lambda_f(X) \neq \Lambda_f(X') \leq (\sum_{\theta\in\Theta_f^k} X(\theta))_{k\in I_f}$, violating the revealed preference. So we must have $\Lambda_f^k(X') \in (\sum_{\theta\in\Theta_f^k} X(\theta), \sum_{\theta\in\Theta_f^k} X'(\theta)]$. We can then define $X^t := tX' + (1-t)X$ and find $t^* \in (0,1]$ such that $\sum_{\theta\in\Theta_f^k} X^{t^*}(\theta) = \Lambda_f^k(X')$. Since $X^{t^*} \leq X'$ and $\Lambda_f(X') \leq (\sum_{\theta\in\Theta_f^k} X^{t^*}(\theta))_{k\in I_f}$, the revealed preference implies $\Lambda_f^k(X^{t^*}) = \Lambda_f^k(X')$, which in turn implies $\rho_f^k(X^{t^*}) = \sum_{\theta\in\Theta_f^k} X^{t^*}(\theta) - \Lambda_f^k(X^{t^*}) = 0 < \sum_{\theta\in\Theta_f^k} X(\theta) - \Lambda_f^k(X) = \rho_f^k(X)$, contradicting the substitutability. *Q.E.D.*

Given Claim S7, it follows that $\alpha_f^k \in [0,1]$ such that

$$\Lambda_f^k(X) = \sum_{\theta\in\Theta_f^k} \min\{X(\theta), \alpha_f^k G(\theta)\},$$

and $\bar{\alpha}_f^k \in [0,1]$ such that

$$\Lambda_f^k(X') = \sum_{\theta\in\Theta_f^k} \min\{X'(\theta), \bar{\alpha}_f^k G(\theta)\},$$

have a relationship $\alpha_f^k \geq \bar{\alpha}_f^k$ (recall $X(\theta) \leq X'(\theta)$ for any $\theta \in \Theta_f^k$ by assumption, and the right-hand side of these equations are nondecreasing in the first arguments of the minimum operators). This implies $\tilde{R}_f(X)(\theta) \leq \tilde{R}_f(X')(\theta)$ for all $\theta \in \Theta_f^k$, as desired. *Q.E.D.*

S.8.2. *Non-Strategy-Proofness for Firms*

Even with a continuum of workers, no stable mechanism is strategy-proof for firms. Consider the following example.[15] Let $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\theta) = G(\theta') = 1/2$. Worker preferences are given as follows:

$$\theta : f_2 \succ f_1 \succ \emptyset,$$
$$\theta' : f_1 \succ f_2 \succ \emptyset.$$

Firm preferences are responsive; f_1 prefers θ to θ' to vacant positions and wants to be matched with workers up to measure 1, while f_2 prefers θ' to θ to vacant positions and wants to be matched with workers up to measure $1/2$.

[15]This example is a continuum-population variant of an example in Section 3 of Hatfield, Kojima, and Narita (2014). See also Azevedo (2014), who showed that stable mechanisms are manipulable via capacities, even in markets with a continuum of workers.

Let φ be any stable mechanism. Given the above input, the following matching is the unique stable matching:

$$M \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta' & \frac{1}{2}\theta \end{pmatrix}.$$

Matching M is clearly stable because it is individually rational and every worker is matched to her most preferred firm. To see the uniqueness, note first that in any stable matching, every worker has to be matched to a firm (if there is a positive measure of unmatched workers, then there is also a vacant position in firm f_1, and they block the matching). All workers of type θ' are matched with f_1; otherwise, f_1 and θ' workers who are not matched with f_1 block the matching (note that f_1 has vacant positions to fill with θ' workers). Given this scenario, all workers of type θ are matched with f_2; otherwise, f_2 and θ workers who are not matched with f_2 block the matching (note that f_2 has vacant positions to fill with type θ workers).

Now, assume that f_1 misreports its preferences, declaring that θ is the only acceptable worker type, and it wants to be matched to them up to measure 1/2. Additionally, assume that preferences of other agents remain unchanged. Then, it is easy to verify that the unique stable matching is

$$M' \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta & \frac{1}{2}\theta' \end{pmatrix}.$$

Therefore, firm f_1 prefers its outcome at M' to the one at M, proving that no stable mechanism is strategy-proof for firms.

S.9. MATCHING WITH CONTRACTS

Our paper has assumed that the terms of employment contracts are exogenously given. In many applications, however, they are decided endogenously. To study such a situation, we generalize our basic model by introducing a continuum-population version of the "matching with contracts" model due to Hatfield and Milgrom (2005).

Let Ω denote a finite set of all available contracts with its typical element denoted as ω. Assume that Ω is partitioned into subsets, $\{\Omega_f\}_{f\in\tilde{F}}$, where Ω_f is the set of contracts for $f \in \tilde{F}$ and $\Omega_\emptyset = \{\omega_\emptyset\}$ (where $\omega_\emptyset$ denotes the option of not contracting with any firm). Each contract ω specifies contract terms a firm f may offer to a worker.[16] Let $f(\omega) \in \tilde{F}$ denote the firm associated with contract ω (or the outside option if $\omega = \omega_\emptyset$). Thus, $f(\omega) = f$ if and only if $\omega \in \Omega_f$. We use $P \in \mathcal{P}$ to denote workers' preference defined over Ω. Let $\omega_-^P \in \Omega$ denote a contract that is an immediate predecessor of ω according to preference P, that is, ω_-^P is the contract with the property $\omega_-^P \succ_P \omega$ and $\omega' \succeq_P \omega_-^P$ for all $\omega' \succ_P \omega$. As before, Θ_P denotes the subset of types in Θ whose preference is given by P.

In the current framework, the relevant unit of analysis is the measure of workers assigned to a particular contract. We let $X_\omega \in \mathcal{X}$ denote the subpopulation assigned to contract $\omega \in \Omega$ and $X_f = (X_\omega)_{\omega\in\Omega_f}$ denote a profile of subpopulations contracting with firm

[16]Note that the contract itself does not contain information about the associated worker type, and that each firm's preference is determined by what worker types it is matched with under what contracts.

f. For any profiles $X, X' \in \mathcal{X}^{|\Omega_f|}$, we denote $X \sqsubset_f X'$ if $X_\omega \sqsubset X'_\omega$ for all $\omega \in \Omega_f$. Given a profile $X_f = (X_\omega)_{\omega \in \Omega_f}$, we use

$$X_f^{\preceq\omega}(\cdot) := \sum_{P \in \mathcal{P}} \sum_{\omega' \in \Omega_f : \omega' \preceq_P \omega} X_{\omega'}(\Theta_P \cap \cdot) \tag{S53}$$

to denote the measure of workers hired by f under contract ω or worse; these are the workers who are willing to work for f under ω given their current contracts. We then let $X_f^{\preceq} = (X_f^{\preceq\omega})_{\omega \in \Omega_f}$.

For any $\omega \in \Omega_f$, let $X_\omega \in \mathcal{X}$ denote the subpopulation of workers who are available to firm f under the contract ω. Given any profile $X_f = (X_\omega)_{\omega \in \Omega_f} \in \mathcal{X}^{|\Omega_f|}$, each firm f's choice is described by a map $X_f \mapsto C_f(X_f) = (C_\omega(X_f))_{\omega \in \Omega_f} \in \mathcal{Y}_f(X_f)$, where

$$\mathcal{Y}_f(X_f) := \{ Y_f \in \mathcal{X}^{|\Omega_f|} \mid Y_f^{\preceq\omega} \sqsubset X_\omega, \forall \omega \in \Omega_f \}.$$

For any profile of subpopulations in $\mathcal{Y}_f(X_f)$, the measure of workers who are hired by f under any contract $\omega \in \Omega_f$ or worse cannot exceed the measure of workers, X_ω, who are available under ω. The requirement that the output of C_f should belong to $\mathcal{Y}_f(X_f)$ is based on the premise that each firm f is aware of workers' preferences and also believes (correctly) that only those workers who are available under $\omega \in \Omega_f$ can be hired under the contracts that are weakly inferior to ω, and thus put an upper bound on the measure of workers that can be hired under the latter contracts. As before, we let $C_{\omega_\emptyset}(X_{\omega_\emptyset}) = X_{\omega_\emptyset}$. We then assume the revealed preference property that for any $X, X' \in \mathcal{X}^{|\Omega_f|}$ with $X' \sqsubset_f X$ and for $M_f = C_f(X)$, if $M_f \in \mathcal{Y}_f(X')$, then $M_f = C_f(X')$.

An **allocation** is $M = (M_\omega)_{\omega \in \Omega}$ such that $M_\omega \in \mathcal{X}$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} M_\omega = G$. Let $M_f = (M_\omega)_{\omega \in \Omega_f} \in \mathcal{X}^{|\Omega_f|}$ denote a profile of subpopulations who are matched with f. Given $M_f = (M_\omega)_{\omega \in \Omega_f}$, define $M_f^{\preceq\omega}$ by (S53) and let $M_f^{\preceq} = (M_f^{\preceq\omega})_{\omega \in \Omega_f}$. Note that $M_f^{\preceq\omega}$ corresponds to a subpopulation of workers already hired by firm f who are willing to work for f under ω given their current contracts. In other words, $M_f^{\preceq}$ does *not* include the workers available to firm f who are currently matched with firms other than f. A subpopulation of *all* workers—not only those hired by firm f—who are available to $f \in \tilde{F}$ under contract $\omega \in \Omega_f$ is denoted as before by

$$D^{\preceq\omega}(M)(\cdot) = \sum_{P \in \mathcal{P}} \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap \cdot).$$

Let $D^{\preceq f}(M) = (D^{\preceq\omega}(M))_{\omega \in \Omega_f}$.

DEFINITION S2: An allocation $M = (M_\omega)_{\omega \in \Omega}$ is **stable** if

1. (Individual Rationality) $M_\omega(\Theta_P) = 0$ for all $P \in \mathcal{P}$ and $\omega \in \Omega$ satisfying $\omega \prec_P \omega_\emptyset$; and for each $f \in F$, $M_f = C_f(M_f^{\preceq})$, and
2. (No Blocking Coalition) there exist no $f \in F$ and $\widetilde{M}_f \in \mathcal{X}^{|\Omega_f|}$, $\widetilde{M}_f \neq M_f$ such that

$$\widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \quad \text{and} \quad \widetilde{M}_f^{\preceq} \sqsubset_f D^{\preceq f}(M).$$

Note that this definition reduces to the notion of stability in Definition 1 if each firm is associated with exactly one contract.

Let us now define a map $T = (T_\omega)_{\omega\in\Omega} : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$ by specifying, for each $\omega \in \Omega$ and $E \in \Sigma$,

$$T_\omega(X)(E) := \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E). \qquad \text{(S54)}$$

THEOREM S1: *$M = (M_\omega)_{\omega\in\Omega}$ is a stable allocation if and only if $M_f = C_f(X_f), \forall f \in \tilde{F}$, where $X = (X_\omega)_{\omega\in\Omega}$ is a fixed point of mapping T.*

PROOF: **("Only if" Part)** Suppose M is a stable allocation in $\mathcal{X}^{|\Omega|}$. We prove that $X = (D^{\preceq\omega}(M))_{\omega\in\Omega}$ is a fixed point of T. Let us first show that for each $\omega \in \Omega$, $X_\omega \in \mathcal{X}$. It is clear that as each M_ω is countably additive, so is $M_\omega(\Theta_P \cap \cdot)$, which implies that $X_\omega(\cdot) = D^{\preceq\omega}(M)(\cdot) = \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap \cdot)$ is also countably additive. It is also clear that since $(M_\omega)_{\omega\in\Omega}$ is an allocation, $X_\omega \sqsubset G$. Thus, we have $X_\omega \in \mathcal{X}$.

We next claim that $M_f = C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \emptyset$ since $M_\emptyset = X_\emptyset = C_\emptyset(X_\emptyset)$. To prove the claim for $f \neq \emptyset$, suppose for a contradiction that $M_f \neq C_f(X_f)$, and let us denote $\widetilde{M}_f = C_f(X_f)$. Since $C_f(X_f) \in \mathcal{Y}_f(X_f)$ by definition, we have $\widetilde{M}_f^{\preceq} \sqsubset_f X_f$ and thus $(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \sqsubset_f X_f$. Given this and $\widetilde{M}_f \in \mathcal{Y}_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$, we have $\widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$ by revealed preference, which means that M is not stable since $\widetilde{M}_f^{\preceq} \sqsubset_f X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X = T(X)$. The fact that $M_\omega = C_\omega(X_{f(\omega)}), \forall\omega \in \Omega$ means that $X_\omega - M_\omega = R_\omega(X_{f(\omega)}), \forall\omega \in \Omega$. Then, for each $\omega \in \Omega$ and $E \in \Sigma$, we obtain

$$\begin{aligned}
&\sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E) \\
&\quad = \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \big(X_{\omega_-^P}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E)\big) \\
&\quad = \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \Bigg(\sum_{\omega'\in\Omega:\omega'\preceq_P\omega_-^P} M_{\omega'}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E) \Bigg) \\
&\quad = \sum_{P:P(1)=\omega} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap E) = X_\omega(E),
\end{aligned}$$

where the second and fourth equalities follow from the definition of $X_{\omega_-^P}$ and X_ω, respectively, while the third follows from the fact that ω_-^P is an immediate predecessor of ω and $\sum_{\omega'\in\Omega:\omega'\preceq_P P(1)} M_{\omega'}(\Theta_P \cap E) = G(\Theta_P \cap E)$. The above equation holds for every contract $\omega \in \Omega$, so we conclude that $X = T(X)$, that is, X is a fixed point of T.

("If" Part) Let us first introduce some notations. Let ω_+^P denote an **immediate successor** of $\omega \in \Omega$ at $P \in \mathcal{P}$: that is, $\omega_+^P \prec_P \omega$, and for any $\omega' \prec_P \omega$, $\omega' \preceq_P \omega_+^P$. Note that for any $\omega, \tilde{\omega} \in \Omega$, $\omega = \tilde{\omega}_-^P$ if and only if $\tilde{\omega} = \omega_+^P$.

Suppose now that $X = (X_\omega)_{\omega\in\Omega} \in \mathcal{X}^{|\Omega|}$ is a fixed point of T. For each contract $\omega \in \Omega$ and $E \in \Sigma$, define

$$M_\omega(E) = X_\omega(E) - \sum_{P:P(|\Omega|)\neq\omega} X_{\omega_+^P}(\Theta_P \cap E), \qquad \text{(S55)}$$

where $P(|\Omega|) \neq \omega$ means that ω is not ranked lowest at P.

We first verify that for each $\omega \in \Omega$, $M_\omega \in \mathcal{X}$. First, it is clear that for each $\omega \in \Omega$, as both $X_\omega(\cdot)$ and $X_{\omega_+^P}(\Theta_P \cap \cdot)$ are countably additive, so is M_ω. It is also clear that for each $\omega \in \Omega$, $M_\omega \sqsubset X_\omega$.

Let us next show that, for all $\omega \in \Omega$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \tag{S56}$$

which means that $X_\omega = D^{\preceq \omega}(M)$. To do so, consider first a contract ω that is ranked lowest at P. By (S55) and the fact that $X_{\omega_+^P}(\Theta_P \cap \cdot) \equiv 0$, we have $M_\omega(\Theta_P \cap E) = X_\omega(\Theta_P \cap E)$. Hence, (S56) holds for such ω. Consider now any $\omega \in \Omega$ which is not ranked last, and assume for an inductive argument that (S56) holds true for ω_+^P, so $X_{\omega_+^P}(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E)$. Then, by (S55), we have

$$\begin{aligned} X_\omega(\Theta_P \cap E) &= M_\omega(\Theta_P \cap E) + X_{\omega_+^P}(\Theta_P \cap E) \\ &= M_\omega(\Theta_P \cap E) + \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E) \\ &= \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \end{aligned}$$

as desired.

To show that $M = (M_\omega)_{\omega \in \Omega}$ is an allocation, let $\omega = P(1)$. Then, the definition of T and the fact that X is a fixed point of T imply that, for any $E \in \Sigma$,

$$G(\Theta_P \cap E) = X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E) = \sum_{\omega' \in \Omega} M_{\omega'}(\Theta_P \cap E),$$

where the second equality follows from (S56). Since the above equation holds for every $P \in \mathcal{P}$, M is an allocation.

We now prove that $(M_\omega)_{\omega \in \Omega}$ is stable. To prove the first part of Condition 1 of Definition S2, note first that $C_{\omega_\emptyset}(X_{\omega_\emptyset}) = \{X_{\omega_\emptyset}\}$ and thus $R_{\omega_\emptyset} = 0$. Fix any $P \in \mathcal{P}$ and assume $\emptyset \neq P(|\Omega|)$, since there is nothing to prove if $\emptyset$ is ranked lowest at P. Consider a contract ω such that $\omega_-^P = \omega_\emptyset$. Then, X being a fixed point of T means $X_\omega(\Theta_P) = R_{\omega_-^P}(\Theta_P) = R_{\omega_\emptyset}(\Theta_P) = 0$, which implies by (S56) that $0 = X_\omega(\Theta_P) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P) = \sum_{\omega' \in \Omega : \omega' \prec_P \omega_\emptyset} M_{\omega'}(\Theta_P)$, as desired.

To prove the second part of Condition 1 of Definition S2, we first show that $M_\omega = C_\omega(X_{f(\omega)})$, which is equivalent to showing $X_\omega - M_\omega = R_\omega(X_{f(\omega)})$. Since $X = T(X)$, we have $X_\omega(\Theta_P \cap \cdot) = R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap \cdot)$ for all $\omega \neq P(1)$, or $X_{\omega_+^P}(\Theta_P \cap \cdot) = R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot)$ for all $\omega \neq P(|\Omega|)$. Then, (S55) implies that, for any $\omega \in \Omega$,

$$X_\omega(\cdot) - M_\omega(\cdot) = \sum_{P : P(|\Omega|) \neq \omega} X_{\omega_+^P}(\Theta_P \cap \cdot) = \sum_{P : P(|\Omega|) \neq \omega} R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) = R_\omega(X_{f(\omega)})(\cdot),$$

as desired. The last equality here follows from the fact that $R_\omega(\Theta_P \cap \cdot) = 0$ if $\omega = P(|\Omega|)$. To see this, note that if $\omega = P(|\Omega|) = \omega_\emptyset$, then $R_\omega(X_{f(\omega)}) = R_{\omega_\emptyset}(X_\emptyset) = 0$ by definition of $R_{\omega_\emptyset}$, and that if $\omega = P(|\Omega|) \prec_P \omega_\emptyset$, then the individual rationality of M for workers implies that $X_\omega(\Theta_P \cap \cdot) = M_\omega(\Theta_P \cap \cdot) = 0$, which in turn implies $R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) = 0$

since $R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) \sqsubset X_\omega(\Theta_P \cap \cdot)$. Given that $M_\omega = C_\omega(X_{f(\omega)})$ for all $\omega \in \Omega$ or $M_f = C_f(X_f)$ for all $f \in F$, $M_f = C_f(M_f^{\precsim})$ follows from the revealed preference and the fact that $M_f^{\precsim} \sqsubset_f X_f$.

It only remains to check Condition 2 of Definition S2. Suppose for a contradiction that it fails. Then, there exist f and $\widetilde{M}_f$ such that

$$M_f \neq \widetilde{M}_f = C_f\big(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim}\big) \quad \text{and} \quad \widetilde{M}_f^{\precsim} \sqsubset_f D^{\preceq f}(M). \tag{S57}$$

Then, we have $M_f \in \mathcal{Y}_f(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim})$, $(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim}) \sqsubset_f D^{\preceq f}(M) = X_f$, and $M_f = C_f(X_f)$, which, by revealed preference, implies $M_f = C_f(\widetilde{M}_f^{\precsim} \vee M_f^{\precsim})$, contradicting (S57). We have thus proven that M is stable. *Q.E.D.*

Given this characterization result, the existence of stable allocation follows from assuming that for each $f \in F$, $C_f : \mathcal{X}^{|\Omega_f|} \to \mathcal{X}^{|\Omega_f|}$ is continuous, since it guarantees the continuity of $T : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$:

THEOREM S2: *If each firm's preference is continuous, then a stable allocation exists.*

S.10. CONTINUUM OF FIRMS: AH MODEL

Following AH, suppose that there is a continuum of firms. Each firm is infinitesimal and takes one of finitely many types, $1, \dots, n$. Let $N = \{1, \dots, n\}$ and $\overline{N} = N \cup \{\emptyset\}$. For each $i \in \overline{N}$, let m_i denote the mass of type-i firms in the economy with $m_\emptyset = \infty$. Assume for simplicity that there are finitely many types of workers so $\Theta = \{\theta_1, \dots, \theta_K\}$. We assume that each type-i firm has a strict preference over the sets in 2^Θ, denoted $\succeq_i$, which gives rise to a choice function $c_i : 2^\Theta \to 2^\Theta$.[17] For a null firm $i = \emptyset$, we let $E \succ_\emptyset E'$ for any $E' \subsetneq E$ and thus $c_\emptyset(E) = E, \forall E \in 2^\Theta$. We assume that $\succeq_i$ satisfies the standard axioms: completeness and transitivity. Each worker can be matched with only one firm (which may be a null firm) and is indifferent over firms of the same type while having strict preferences over different types of firms. We denote this economy as $\mathcal{E}$. This model is exactly the same as AH, except that there is no contracting issue (a firm and worker can contract under only one term) and we are considering a many-to-one matching environment.

A matching for type-i firms is a measure z_i defined on 2^Θ such that, for each $E \in 2^\Theta$, $z_i(E)$ is the measure (or mass) of type-i firms matched with E. A profile $(z_i)_{i \in \overline{N}}$ is a matching if

$$\sum_{i \in \overline{N}} \sum_{E \in 2^\Theta : \theta \in E} z_i(E) = G(\theta), \quad \forall \theta \in \Theta, \tag{S58}$$

$$\sum_{E \in 2^\Theta} z_i(E) = m_i, \quad \forall i \in N. \tag{S59}$$

DEFINITION S3: A matching $z = (z_i)_{i \in \overline{N}}$ is stable for the economy $\mathcal{E}$ if the following properties hold:

[17] An implicit assumption here is that each firm hires at most one worker per each worker type. However, our model can be extended in a straightforward manner to allow each firm to hire multiple workers of the same type.

1. (Individual Rationality) $z_i(E) = 0$ for any $i \in N$ and $E \in 2^\Theta$ such that there is some $\theta \in E$ with $\emptyset \succ_\theta i$; for any $i \in N$ and $E \in 2^\Theta$, $z_i(E) > 0$ implies $c_i(E) = E$;
2. (No Blocking Coalition) there are no $i \in N$ and $E, E' \in 2^\Theta$ with $E \cap E' = \emptyset$ such that (i) $E' \subset c_i(E \cup E')$; (ii) $z_i(E) > 0$; and (iii) for each $\theta \in E'$, there are $j \in \overline{N}$ and $E'' \in 2^\Theta$ such that $i \succ_\theta j$, $\theta \in E''$, and $z_j(E'') > 0$.

Individual rationality condition is straightforward. No blocking coalition condition requires no positive mass of firms which can get better off by hiring workers away from their less preferred firms. This notion of stability coincides with that of AH, once their model of many-to-many matching with contracts is adapted to our setup.

To show the existence of stable matching, we map the current setting into our model of continuum economy by introducing a large firm representing all type-i firms for each type $i \in \overline{N}$ and defining the *aggregate choice correspondence* for this firm, denoted $C_i : \mathcal{X} \rightrightarrows \mathcal{X}$. To do so, suppose that $X_i \in \mathcal{X}$ is a subpopulation of workers available to the large type-i firm, which is a subpopulation defined on Θ. We then allocate these workers *efficiently* across type-i firms as follows: Endow each small type-i firm with an arbitrary utility function $v_i : 2^\Theta \to \mathbb{R}_+$ that represents $\succeq_i$ and satisfies $v_i(\emptyset) = 0$.[18] And assign a set of workers $E \subset \Theta$ to the mass $z_i(E)$ of type-i firms for each $E \in 2^\Theta$ to solve

$$\max_{z_i \in \mathbb{R}_+^{|2^\Theta|}} \sum_{E \in 2^\Theta} v_i(E) z_i(E) \tag{A}$$

subject to

$$\sum_{E' \in 2^\Theta : \theta \in E'} z_i(E') \le X_i(\theta), \quad \forall \theta \in \Theta, \tag{S60}$$

$$\sum_{E \in 2^\Theta} z_i(E) = m_i, \tag{S61}$$

where the constraint (S61) is dropped if $i = \emptyset$.[19] That is, the aggregate (utilitarian) welfare of type-i firms is maximized under the constraint that for each type θ, the measure of type-i firms hiring (some) type-θ workers cannot exceed the measure of available type-θ workers. Letting $S_i(X_i)$ denote the set of optimal solutions for (A), it is straightforward to see that $S_i(X_i)$ is nonempty.

The aggregate choice correspondence for the *large firm i* is then defined as

$$C_i(X_i) = \left\{ X_i' \in \mathcal{X} | \exists z_i \in S_i(X_i) \text{ such that } X_i'(\theta) = \sum_{E' \in 2^\Theta : \theta \in E'} z_i(E'), \forall \theta \in \Theta \right\}.^{20}$$

It is worth noting that our method to build the aggregate choice differs from that of AH in which firms of the same type choose workers following *serial dictatorship*. We let Γ denote a hypothetical economy that consists of large firms $1, \ldots, n, \emptyset$, whose choice correspondences are given as $(C_i)_{i \in \overline{N}}$, and workers whose population is given as G. Since (A) is

[18]Existence of such v_i is guaranteed because the firms' preferences satisfy the standard axioms.

[19]Recall that $m_\emptyset = \infty$. Note that the constraint (S60) must always be binding for $i = \emptyset$ at any optimum since $v_\emptyset(E) > v_\emptyset(\emptyset) = 0$ for any $E \neq \emptyset$, as implied by the earlier assumption.

[20]Since each $S_i(X_i)$ consists of optimal solutions, S_i satisfies the revealed preference. Given this, C_i also satisfies the revealed preference property.

linear, and thus continuous, in z_i, by Berge's maximum theorem, each correspondence S_i is upper hemicontinuous and convex-valued, so is C_i. Hence, by Theorem 2, there exists a stable matching in economy Γ, which implies the existence of a stable matching in the original economy $\mathcal{E}$, as is shown next:

PROPOSITION S5: *Let $M = (M_i)_{i\in\overline{N}}$ be a stable matching for the hypothetical economy Γ. Then, there is a profile of solutions $z = (z_i)_{i\in\overline{N}}$ for (A) with $X_i = M_i$, $\forall i \in \overline{N}$ that constitutes a stable matching for economy $\mathcal{E}$.*

PROOF: First, there must be a solution of (A) with $X_i = M_i$ that satisfies (S60) as equality, since otherwise M_i would not be individually rational in economy Γ. Now let $z = (z_i)_{i\in\overline{N}}$ be a profile of such solutions. First of all, we check that z is a matching in economy $\mathcal{E}$. That (S60) is binding with $X_i(\theta) = M_i(\theta)$ implies (S58) is satisfied since M is a matching so $\sum_{i\in\overline{N}} M_i(\theta) = G(\theta)$. Also, (S59) follows directly from (S61).

Note next that since M is stable in economy Γ, we must have $M_i \in C_i(\tilde{X}_i)$ for $\tilde{X}_i = D^{\preceq i}(M)$, which implies that $(z_i)_{i\in\overline{N}}$ solves (A) with $X_i = \tilde{X}_i$.

To show the stability of z in economy $\mathcal{E}$, we first prove that it is individually rational. To see the individual rationality for workers, observe that, for any $\theta \in \Theta$ and $\emptyset \succ_\theta i$, we have $M_i(\theta) = 0$, which follows from the stability of M in economy Γ. It therefore follows from (S60) with $X_i(\theta) = M_i(\theta)$ that $z_i(E) = 0$ for any E containing θ. To see individual rationality of $(z_i)_{i\in\overline{N}}$ for firms, supppose not. Then, there must be some firm $i \in N$ and $E \in 2^\Theta$ such that $z_i(E) > 0$ and $c_i(E) \subsetneq E$, which means that $v_i(E) < v_i(c_i(E))$. Given this, consider another matching for type-i firms which assigns the set of workers $c_i(E)$ to the type-i firms of mass $z_i(E)$ which are hiring E under z_i, while assigning the same set of workers to all other type-i firms in N. This alternative matching then achieves a higher value for (A), which contradicts the optimality of z_i.

We next prove z satisfies the second requirement of stability in economy $\mathcal{E}$. Suppose for contradiction that $(z_i)_{i\in\overline{N}}$ admits a blocking coalition with the firm type i and E, E' as in Condition 2 of Definition S3. Let, for each $\theta \in E'$,

$$\bar{z}(\theta) = \max\{z_j(E'')|j \in \overline{N}, \theta \in E'', z_j(E'') > 0, \text{ and } i \succ_\theta j\}.$$

Then, at least $\bar{z}(\theta)$ of workers of type $\theta \in E'$ are not matched with type-i firms under $(z_i)_{i\in\overline{N}}$ but available to them under $\tilde{X}_i = D^{\preceq i}(M)$. Consider now an alternative matching z_i' for type-i firms given as follows: (1) mass $\min\{\min_{\theta\in E''} \bar{z}(\theta), z_i(E)\}$ of type-i firms which were matched with E under z_i are now each matched with the set $c_i(E \cup E')$ of workers; (2) all other type-i firms are matched with the same set of workers as under z_i. Note first that the workers matched with type-i firms under z_i' are a subpopulation of $\tilde{X}_i$, satisfying (S60) with $X_i = \tilde{X}_i$. Also, z_i' easily satisfies (S61). However, since $c_i(E \cup E') \neq E$, we have $v_i(c_i(E \cup E')) > v_i(E)$, which means that the type-i firms in (1) above enjoy a higher utility under z_i' than z_i, while the type-i firms in (2) enjoy the same utility. This contradicts the fact that $(z_i)_{i\in\overline{N}}$ solves (A) with $X_i = \tilde{X}_i$. *Q.E.D.*

COROLLARY S1: *There exists a stable matching for economy $\mathcal{E}$.*

Recall that the approach taken here to build the aggregate choice correspondence differs from that of AH based on the serial dictatorship. One advantage of the current approach is its extendability beyond finite types of workers. It is not difficult to extend (A)

to allow for continuum of worker types. Since (A) is linear, its solution set (or correspondences) will satisfy the properties such as upper hemicontinuity and convex-valuedness (as long as v_i is a continuous function).[21]

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[21] In the functional space, a linearity need not imply continuity. But in our case, as long as v_i is assumed to be continuous, the objective function of (A) is continuous in z_i in the weak-$*$ topology.