*Econometrica Supplementary Material*

# SUPPLEMENT TO "MARKET MICROSTRUCTURE INVARIANCE: EMPIRICAL HYPOTHESES"

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## CURRENT RESEARCH ON INVARIANCE HYPOTHESES

THE INVARIANCE HYPOTHESES CAN BE EXAMINED using many different data sets. Here is a summary of our current research using data sets available to academic researchers:

- The Ancerno data set includes more orders than the data set of portfolio transitions used in this paper. The data set groups trades into "meta-orders" which approximate our concept of a bet. Preliminary research by Albert S. Kyle and Kingsley Fong found that proxies for bets in Ancerno data have size patterns consistent with the invariance hypotheses. The Ancerno data are also designed to facilitate measurement of transactions costs. This data set is therefore appropriate for validating our empirical results for both bet size and transaction costs.
- Andersen, Bondarenko, Kyle, and Obizhaeva (2015) examined the variation in trade frequency, contract volume, and volatility in the S&P 500 E-mini futures contract across minutes of the 24-hour day. The results conform to predicted invariance relationships.

TABLE A.I

QUANTILE ESTIMATES OF ORDER SIZE[a]

| | p1 | p5 | p25 | p50 | p75 | p95 | p99 |
|---|---|---|---|---|---|---|---|
| $\ln[\bar{q}]$ | −9.37 | −8.31 | −6.73 | −5.66 | −4.59 | −3.05 | −2.05 |
| | (0.008) | (0.006) | (0.004) | (0.003) | (0.004) | (0.006) | (0.009) |
| $\alpha_0$ | −0.65 | −0.64 | −0.61 | −0.62 | −0.61 | −0.64 | −0.63 |
| | (0.005) | (0.003) | (0.002) | (0.002) | (0.002) | (0.003) | (0.005) |
| Pseudo $R^2$ | 0.1621 | 0.1534 | 0.1650 | 0.1727 | 0.1795 | 0.1949 | 0.2232 |
| $Q^*/V^* \cdot \delta \times 10^4$ | 0.85 | 2.46 | 11.95 | 34.83 | 101.53 | 473.59 | 1287.35 |
| #Obs | 439,765 | 439,765 | 439,765 | 439,765 | 439,765 | 439,765 | 439,765 |

[a]Table presents the estimates $\ln[\bar{q}]$ and $\alpha_0$ for the quantile regression

$$\ln\left[\frac{X_i}{V_i}\right] = \ln[\bar{q}] + \alpha_0 \cdot \ln\left[\frac{W_i}{W^*}\right] + \tilde{\varepsilon}_i.$$

Each observation corresponds to transition order $i$ with order size $X_i$, pre-transition price $P_i$, expected daily volume $V_i$, expected daily volatility $\sigma_i$, trading activity $W_i$. The parameter $\bar{q}$ is the measure of order size such that for $\delta = 1$, $Q^*/V^* \cdot \delta^{-1} \times 10^4$ measures the median bet size for the benchmark stock, in basis points of average daily volume. The benchmark stock has daily volatility of 2%, share price of $40, and daily volume of one million shares. The standard errors are shown in parentheses. The sample ranges from January 2001 to December 2005.

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TABLE A.II

OLS ESTIMATES FOR ORDER SIZE: MODEL CALIBRATION[a]

| | | NYSE | | NASDAQ | |
|---|---|---|---|---|---|
| | All | Buy | Sell | Buy | Sell |
| | *Restricted Specification*: $\alpha_0 = -2/3$, $b_1 = b_2 = b_3 = b_4 = 0$ | | | | |
| $\ln[\bar{q}]$ | −5.71 | −5.70 | −5.68 | −5.70 | −5.77 |
| | (0.019) | (0.023) | (0.019) | (0.042) | (0.039) |
| $Q^*/V^* \cdot \delta \times 10^4$ | 33.13 | 33.46 | 34.14 | 33.46 | 31.20 |
| MSE | 2.53 | 2.61 | 2.54 | 2.32 | 2.56 |
| $R^2$ | 0.3149 | 0.2578 | 0.2599 | 0.4278 | 0.3479 |
| | *Unrestricted Specification With 5 Degrees of Freedom*: $\alpha_0 = -2/3$ | | | | |
| $\ln[\bar{q}]$ | −5.53 | −5.55 | −5.48 | −5.77 | −5.48 |
| | (0.019) | (0.026) | (0.019) | (0.051) | (0.047) |
| $b_1$ | 0.42 | 0.47 | 0.53 | 0.19 | 0.33 |
| | (0.040) | (0.050) | (0.043) | (0.094) | (0.087) |
| $b_2$ | 0.24 | 0.17 | 0.29 | 0.04 | 0.33 |
| | (0.019) | (0.021) | (0.017) | (0.049) | (0.040) |
| $b_3$ | 0.06 | 0.06 | 0.07 | −0.06 | 0.07 |
| | (0.010) | (0.012) | (0.009) | (0.026) | (0.021) |
| $b_4$ | −0.18 | −0.24 | −0.22 | −0.02 | −0.11 |
| | (0.015) | (0.020) | (0.017) | (0.040) | (0.032) |
| $R^2$ | 0.3229 | 0.2668 | 0.2739 | 0.4318 | 0.3616 |
| #Obs | 439,765 | 131,530 | 150,377 | 69,871 | 87,987 |

[a]Table presents the estimates and the mean squared error (MSE) for the regression

$$\ln\left[\frac{X_i}{V_i}\right] = \ln[\bar{q}] + \alpha_0 \cdot \ln\left[\frac{W_i}{W^*}\right] + b_1 \cdot \ln\left[\frac{\sigma_i}{0.02}\right] + b_2 \cdot \ln\left[\frac{P_i}{40}\right] + b_3 \cdot \ln\left[\frac{V_i}{10^6}\right] + b_4 \cdot \ln\left[\frac{\nu_i}{1/12}\right] + \tilde{\varepsilon}_i,$$

with $\alpha_0$ restricted to be $-2/3$ as predicted by invariance and $b_1 = b_2 = b_3 = 0$. Each observation corresponds to transition order $i$ with order size $X_i$, pre-transition price $P_i$, expected daily volume $V_i$, expected daily volatility $\sigma_i$, trading activity $W_i$, and monthly turnover rate $\nu_i$. The parameter $\bar{q}$ is the measure of order size such that for $\delta = 1$, $Q^*/V^* \cdot \delta^{-1} \times 10^4$ measures the median bet size for the benchmark stock, in basis points of average daily volume. The benchmark stock has daily volatility of 2%, share price of \$40, and daily volume of one million shares. The $R^2$'s are reported for restricted specification with $\alpha_0 = -2/3$, $b_1 = b_2 = b_3 = b_4 = 0$ as well as for unrestricted specification with coefficients $\ln[\bar{q}]$ and $b_1, b_2, b_3, b_4$ allowed to vary freely. The standard errors are clustered at weekly levels for 17 industries and shown in parentheses. The sample ranges from January 2001 to December 2005.

- Kyle and Obizhaeva (2016) compared extrapolations of the linear market impact estimates in this paper to publicly documented quantities sold in five stock market crashes. The price declines in the 1987 crash and the 2008 liquidation of Jerome Kerviel's rogue trades at Societe Generale, which occurred over time frames similar to large portfolio transitions, were close to the predicted declines. Transitory price declines were larger than predicted in the two "flash crashes," when sales occurred unusually rapidly, and smaller in the 1929 crash, when sales were stretched out over weeks and months. While consistent with the invariance hypotheses, these results also suggest that the speed of execution influences temporary market impact.

TABLE A.III

TRANSACTION-COST ESTIMATES IN REGRESSION WITH LINEAR IMPACT[a]

| | | NYSE | | NASDAQ | |
|---|---|---|---|---|---|
| | All | Buy | Sell | Buy | Sell |
| $\beta_{\text{mkt}}$ | 0.66 | 0.63 | 0.62 | 0.77 | 0.77 |
| | (0.013) | (0.016) | (0.016) | (0.037) | (0.036) |
| $\kappa_0^* \times 10^4$ | 6.28 | 6.51 | 5.43 | 5.94 | 6.54 |
| | (0.890) | (1.600) | (1.154) | (2.147) | (1.501) |
| $\alpha_1$ | −0.40 | −0.36 | −0.39 | −0.44 | −0.40 |
| | (0.020) | (0.048) | (0.029) | (0.051) | (0.031) |
| $\kappa_I^* \times 10^4$ | 2.73 | 2.63 | 2.10 | 3.69 | 3.13 |
| | (0.252) | (0.460) | (0.346) | (0.663) | (0.765) |
| $\alpha_2$ | −0.31 | −0.45 | −0.31 | −0.32 | −0.28 |
| | (0.028) | (0.038) | (0.041) | (0.056) | (0.058) |
| $R^2$ | 0.0993 | 0.1105 | 0.1014 | 0.0931 | 0.0901 |
| #Obs | 439,765 | 131,530 | 150,377 | 69,871 | 87,987 |

[a]Table presents the estimates for $\beta_{\text{mkt}}, \alpha_1, \kappa_0^*, \alpha_2$, and $\kappa_I^*$ in the regression

$$\mathbb{I}_{\text{BS},i} \cdot S_i \cdot \frac{(0.02)}{\sigma_i} = \beta_{\text{mkt}} \cdot R_{\text{mkt},i} \cdot \frac{(0.02)}{\sigma_i} + \mathbb{I}_{\text{BS},i} \cdot \kappa_0^* \cdot \left[\frac{W_i}{W^*}\right]^{\alpha_1} + \mathbb{I}_{\text{BS},i} \cdot \kappa_I^* \cdot \left[\frac{W_i}{W^*}\right]^{\alpha_2} \cdot \left[\frac{\phi I_i}{0.01}\right]^z + \tilde{\varepsilon}_i,$$

where $z = 1$ and $\phi I_i/0.01 = X_i/(0.01V_i) \cdot (W_i/W^*)^{2/3}$. $S_i$ is implementation shortfall. $R_{\text{mkt},i}$ is the value-weight market return for the first day of transition. The trading activity $W_i$ is the product of expected volatility $\sigma_i$, pre-transition price $P_i$, and expected volume $V_i$. The scaling constant $W^* = (0.02)(40)(10^6)$ is the trading activity for the benchmark stock with volatility of 2% per day, price $40 per share, and trading volume of one million shares per day. $X_i$ is the number of shares in the order $i$. The parameter $\kappa_I^* \times 10^4$ is the market impact cost of executing a trade of one percent of daily volume in the benchmark stock, and $\kappa_0^* \times 10^4$ is the effective spread cost; both are measured in basis points. The standard errors are clustered at weekly levels for 17 industries and shown in parentheses. The sample ranges from January 2001 to December 2005.

- Kyle, Obizhaeva, and Tuzun (2016) examined the hypothesis that the size of "prints" of stock market trades in Trade and Quotations (TAQ) data is proportional to the size of bets. This hypothesis holds up well during the 1990s, consistent with the interpretation that bets were negotiated and printed as block trades. The hypothesis breaks down after 2001, when trade size collapsed toward the round-lot minimum size of 100 shares for many trades. The changes after 2001 may be the result of the reduction in the minimum tick size to one cent and the increased use of electronic order-shredding algorithms.
- Kyle, Obizhaeva, and Sinha (2012) examined whether the monthly frequency of Thomson Reuters news articles is proportional to the 2/3 power of trading activity for individual stocks. The estimated exponent is close to the predicted value of 2/3 prior to a strategic decision by Thomson Reuters to increase the number of news articles about less actively traded stocks, after which the coefficient changes in a manner consistent with their strategic decision.

Using a different proprietary data set, Bae, Kyle, Lee, and Obizhaeva (2014) examined the number of times individual trading accounts switch between buying and selling individual stocks. Consistent with the predictions of invariance,

TABLE A.IV

TRANSACTION COSTS: MODEL CALIBRATION[a]

| | | NYSE | | NASDAQ | |
|---|---|---|---|---|---|
| | All | Buy | Sell | Buy | Sell |
| | *Linear Model*: $z = 1$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$ | | | | |
| $\beta_{\text{mkt}}$ | 0.6571 | 0.6308 | 0.6195 | 0.7693 | 0.7771 |
| | (0.0135) | (0.0159) | (0.0158) | (0.0371) | (0.0365) |
| $\kappa_0 \times 10^4$ | 8.2134 | 7.1934 | 6.7698 | 9.1832 | 9.2658 |
| | (0.5776) | (1.1215) | (0.7943) | (1.5627) | (0.7811) |
| $\kappa_I \times 10^4$ | 2.5003 | 3.3663 | 1.9220 | 3.4614 | 2.4629 |
| | (0.1903) | (0.3700) | (0.2650) | (0.3953) | (0.3267) |
| $R^2$ | 0.0991 | 0.1102 | 0.1012 | 0.0926 | 0.0897 |
| | *Square-Root Model*: $z = 1/2$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = \beta_7 = \beta_8 = 0$ | | | | |
| $\beta_{\text{mkt}}$ | 0.6552 | 0.6285 | 0.6192 | 0.7598 | 0.7782 |
| | (0.0134) | (0.0158) | (0.0159) | (0.0365) | (0.0364) |
| $\kappa_0 \times 10^4$ | 2.0763 | −1.3091 | 0.9167 | 2.2844 | 4.6530 |
| | (0.7035) | (1.2779) | (0.9264) | (2.0554) | (0.8244) |
| $\kappa_I \times 10^4$ | 12.0787 | 15.6544 | 11.0986 | 13.5025 | 10.4063 |
| | (0.7416) | (1.2177) | (1.2979) | (1.4564) | (1.2069) |
| $R^2$ | 0.1007 | 0.1116 | 0.1027 | 0.0941 | 0.0911 |

(*Continues*)

TABLE A.IV—*Continued*

| | All | NYSE Buy | NYSE Sell | NASDAQ Buy | NASDAQ Sell |
|---|---|---|---|---|---|
| | *Unrestricted Specification With 12 Degrees of Freedom* | | | | |
| $\beta_{\text{mkt}}$ | 0.66 | 0.63 | 0.62 | 0.76 | 0.78 |
| | (0.013) | (0.016) | (0.015) | (0.036) | (0.036) |
| $\kappa_0^* \times 10^4$ | 0.94 | −0.05 | 0.47 | 1.55 | 1.61 |
| | (0.675) | (0.124) | (0.556) | (1.698) | (1.148) |
| $\beta_1$ | −0.43 | −2.47 | −1.08 | −0.44 | −0.46 |
| | (0.147) | (0.890) | (0.392) | (0.489) | (0.131) |
| $\beta_2$ | 0.17 | 2.87 | 0.23 | 0.20 | 0.11 |
| | (0.072) | (1.230) | (0.231) | (0.127) | (0.109) |
| $\beta_3$ | −0.56 | 1.85 | −0.47 | −0.47 | −0.49 |
| | (0.159) | (0.754) | (0.296) | (0.238) | (0.155) |
| $\beta_4$ | 0.62 | 0.13 | 0.49 | 0.49 | 0.58 |
| | (0.173) | (0.620) | (0.490) | (0.313) | (0.175) |
| $\kappa_I^* \times 10^4$ | 9.36 | 11.61 | 10.93 | 8.88 | 5.00 |
| | (1.307) | (2.471) | (1.804) | (3.340) | (2.033) |
| $z$ | 0.58 | 0.54 | 0.52 | 0.58 | 0.63 |
| | (0.041) | (0.039) | (0.042) | (0.094) | (0.083) |
| $\beta_5$ | 0.02 | −0.11 | 0.36 | −0.17 | −0.23 |
| | (0.135) | (0.192) | (0.229) | (0.252) | (0.242) |
| $\beta_6$ | −0.14 | −0.11 | 0.03 | −0.27* | −0.22 |
| | (0.061) | (0.113) | (0.100) | (0.120) | (0.113) |
| $\beta_7$ | 0.01 | −0.07 | 0.04 | 0.00 | −0.16 |
| | (0.037) | (0.050) | (0.052) | (0.099) | (0.100) |
| $\beta_8$ | 0.08 | 0.07 | −0.11 | 0.08 | 0.39 |
| | (0.067) | (0.086) | (0.101) | (0.143) | (0.153) |
| $R^2$ | 0.1016 | 0.1121 | 0.1032 | 0.0957 | 0.0944 |
| #Obs | 439,765 | 131,530 | 150,377 | 69,871 | 87,987 |

[a]Table presents the estimates for the regression

$$\mathbb{I}_{\text{BS},i} \cdot S_i \cdot \frac{(0.02)}{\sigma_i} = \beta_{\text{mkt}} \cdot R_{\text{mkt},i} \cdot \frac{(0.02)}{\sigma_i} \mathbb{I}_{\text{BS},i} \cdot \kappa_0^* \cdot \left[\frac{W_i}{W^*}\right]^{-1/3} \cdot \frac{\sigma_i^{\beta_1} \cdot P_i^{\beta_2} \cdot V_i^{\beta_3} \cdot \nu_i^{\beta_4}}{(0.02)(40)(10^6)(1/12)}$$

$$+ \mathbb{I}_{\text{BS},i} \cdot \kappa_I^* \cdot \left[\frac{\phi I_i}{0.01}\right]^z \cdot \left[\frac{W_i}{W^*}\right]^{-1/3} \cdot \frac{\sigma_i^{\beta_5} \cdot P_i^{\beta_6} \cdot V_i^{\beta_7} \cdot \nu_i^{\beta_8}}{(0.02)(40)(10^6)(1/12)} + \tilde{\varepsilon}_i,$$

where $\phi I_i/0.01 = X_i/(0.01V_i) \cdot (W_i/W^*)^{2/3}$. $S_i$ is implementation shortfall. $R_{\text{mkt},i}$ is the value-weight market return for the first day of transition. The trading activity $W_i$ is the product of expected volatility $\sigma_i$, pre-transition price $P_i$, and expected volume $V_i$. The scaling constant $W^* = (0.02)(40)(10^6)$ is the trading activity for the benchmark stock with volatility of 2% per day, price $40 per share, and trading volume of one million shares per day. $X_i$ is the number of shares in the order $i$. The parameter $\kappa_I^* \times 10^4$ is the market impact cost of executing a trade of one percent of daily volume in the benchmark stock, and $\kappa_0^* \times 10^4$ is the effective spread cost; both are measured in basis points. The $R^2$'s are reported for restricted specification as well as for unrestricted specification with twelve coefficients $\beta_{\text{mkt}}$, $z$, $\kappa_I^*$, $\kappa_0^*$, $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$ allowed to vary freely. The standard errors are clustered at weekly levels for 17 industries and shown in parentheses. The sample ranges from January 2001 to December 2005.

TABLE A.V

TRANSACTION-COST ESTIMATES IN REGRESSION WITH QUOTED SPREAD[a]

| | | NYSE | | NASDAQ | |
|---|---|---|---|---|---|
| | All | Buy | Sell | Buy | Sell |
| $\beta_{\text{mkt}}$ | 0.65 | 0.63 | 0.62 | 0.76 | 0.77 |
| | (0.013) | (0.016) | (0.015) | (0.036) | (0.037) |
| $\kappa_I \times 10^4$ | 2.95 | 2.97 | 2.24 | 3.76 | 2.95 |
| | (0.261) | (0.504) | (0.366) | (0.700) | (0.749) |
| $\alpha_2$ | −0.32 | −0.44 | −0.32 | −0.37 | −0.33 |
| | (0.029) | (0.036) | (0.039) | (0.053) | (0.060) |
| $\beta_S$ | 0.71 | 0.61 | 0.74 | 0.61 | 0.75 |
| | (0.053) | (0.110) | (0.094) | (0.127) | (0.073) |
| $R^2$ | 0.0976 | 0.1094 | 0.1010 | 0.0891 | 0.0872 |
| #Obs | 436,649 | 131,100 | 149,600 | 69,218 | 86,731 |

[a]Table presents the estimates for $\beta_{\text{mkt}}$, $\kappa_I$, $\alpha_2$, and $\beta_S$ in the regression

$$\mathbb{I}_{\text{BS},i} \cdot S_i \cdot \frac{(0.02)}{\sigma_i} = \beta_{\text{mkt}} \cdot R_{\text{mkt},i} \cdot \frac{(0.02)}{\sigma_i} + \mathbb{I}_{\text{BS},i} \cdot \beta_S \cdot \frac{1}{2} \cdot \frac{s_i}{P_i} \cdot \frac{(0.02)}{\sigma_i}$$
$$+ \mathbb{I}_{\text{BS},i} \cdot \kappa_I \cdot \left[\frac{\phi I_i}{0.01}\right] \cdot \left[\frac{W_i}{W^*}\right]^{\alpha_2} + \tilde{\varepsilon}_i,$$

where invariant $I_i = \frac{X_i}{(0.01)V_i} \cdot [\frac{W_i}{W^*}]^{2/3}$. Each observation corresponds to order $i$. $\mathbb{I}_{\text{BS},i}$ is a buy/sell indicator, $S_i$ is implementation shortfall, $R_{\text{mkt},i}$ is the value-weight market return for the first day of transition. The term $(0.02)/\sigma_i$ adjusts for heteroscedasticity. The trading activity $W_i$ is the product of expected volatility $\sigma_i$, pre-transition price $P_i$, and expected volume $V_i$. The scaling constant $W^* = (0.02)(40)(10^6)$ is the trading activity for the benchmark stock with volatility of 2% per day, price $40 per share, and trading volume of one million shares per day. $X_i$ is the number of shares in the order $i$. The parameter $\kappa_I^* \times 10^4$ is the market impact cost of executing a trade of one percent of daily volume in the benchmark stock, measured in basis points. $s_i/P_i$ is the quoted percentage spread. The standard errors are clustered at weekly levels for 17 industries and shown in parentheses. The sample ranges from January 2001 to December 2005.

it is shown that the number of switching points is proportional to the 2/3 power of trading activity.

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