Econometrica Supplementary Material

SUPPLEMENT TO "TOWARD A STRATEGIC FOUNDATION FOR RATIONAL EXPECTATIONS EQUILIBRIUM"

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The proof of the main result in the main paper is provided here. We also provide an example in which the best reply to nondecreasing bidding functions fails to be nondecreasing, and we show how to approximate a degenerate density by one that satisfies the assumptions in the main paper. Finally, we establish that the main result continues to hold when the notion of genericity is changed from the topological notion of residual sets to the measure-motivated notion of prevalent sets.

1. INTRODUCTION

THROUGHOUT THIS SUPPLEMENT, we refer to the main paper as RP. This supplement proceeds as follows: Section 2 provides the proof of RP Theorem 6.1. Section 3 provides an example in which one agent has no nondecreasing best reply to the nondecreasing bidding functions of the other agents. Section 4 provides a sequence of conditional density functions that satisfy RP Assumptions A.1 and A.2, which converge uniformly to the density function employed in the example in RP Section 5.2. Section 5 proves that RP Theorem 6.1 continues to hold when the topologically motivated notion of genericity used in RP (residual sets) is replaced by a measure-motivated notion of genericity (prevalent sets).

2. PROOF OF THEOREM 6.1 IN RP

The proof is broken into Parts A–D.

2.1. *Part A*

We begin by demonstrating that when there is a continuum of agents and the grid of prices is fine enough, the double auction possesses a symmetric equilibrium in pure nondecreasing bidding functions.

As in RP Section 3, suppose that there is a unit mass of agents, of whom $\alpha \in (0,1)$ are buyers and $1-\alpha$ are sellers. If the state of the good is ω, drawn according to the density $g(\cdot)$, then for every $x \in [0,1]$, $F(x|\omega)$ agents receive a signal below x and $1-F(x|\omega)$ agents receive a signal above x. An agent with signal x has value $v(x,\omega)$ in state ω. Now suppose that each agent's bid is restricted to a discrete set of prices $\mathcal{P} = \{0, \Delta, 2\Delta, \ldots\}$, where Δ is the fineness

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of the grid. Denote this environment by $\mathcal{E}(\alpha, v, f, g, \Delta)$, where f is the density of F.

Let us now define a pure, symmetric, nondecreasing double-auction equilibrium for $\mathcal{E}(\alpha, v, f, g, \Delta)$. We restrict our attention to nondecreasing bidding functions that are right-continuous at every $x \in [0, 1]$ and continuous at $x = 1$. This is convenient because such functions are completely characterized by their jump points and every bid in the range of such a function is assumed on a nondegenerate interval of signals. So, fix such a nondecreasing function $b : [0, 1] \to \mathcal{P}$.

We wish to define an agent's payoff as a function of both his signal and his bid given that all other agents employ $b : [0, 1] \to \mathcal{P}$, but to do so, we must first determine the market-clearing price as a function of the unknown state of the good ω when all other agents employ $b(\cdot)$. Recall that $x(\omega)$ is the αth percentile when the state of the good is ω (i.e., $F(x(\omega)|\omega) = \alpha$). Hence, if all agents employ $b(\cdot)$, then the double-auction market-clearing price in state ω must be $P(\omega) = b(x(\omega))$ whenever $b(\cdot)$ is continuous at $x(\omega)$. Because $x(\cdot)$ is strictly increasing, $b(\cdot)$, being nondecreasing, is continuous at $x(\omega)$ for all but perhaps countably many ω. Consequently, $P(\omega)$ is determined uniquely for all but perhaps countably many, and hence a measure zero set of, states ω. Therefore, without loss, we may define $P(\omega) = b(x(\omega))$ for all $\omega \in [0, 1)$ and define $P(1) = \lim_{\omega \uparrow 1} P(\omega)$, which is well defined because $b(x(\cdot))$ is nondecreasing.[2] Henceforth, we will abbreviate this definition of $P(\cdot)$, including the limit at $\omega = 1$, by writing $P(\cdot) \equiv b(x(\cdot))$.

Suppose that $p = P(\omega)$ is the market-clearing price in state $\omega \in [0, 1]$. Letting $[\underline{x}(p), \bar{x}(p))$ denote the nondegenerate interval in $[0, 1)$ on which $b(\cdot)$ is p, $F(\bar{x}(p)|\omega) - F(\underline{x}(p)|\omega)$ is the fraction of agents who bid p; the mass of agents who bid p or higher is $1 - F(\underline{x}(p)|\omega) \geq 1 - F(x(\omega)|\omega) = 1 - \alpha$. However, but because $1 - \alpha$ is the total number of units of the good, only a fraction of the agents who bid p will end up with a unit.

Because the mass of agents who bid strictly more than p in state ω, namely $1 - F(\bar{x}(p)|\omega)$, must end up with a unit, clearing the market requires that, out of those agents whose bids are equal to the market-clearing price p, only $F(\bar{x}(p)|\omega) - \alpha$ end up with a unit. Since these agents are chosen at random from among those who bid p, an agent who bids p when the state is $\omega \in P^{-1}(p)$ ends up with a unit with probability

$$\lambda(\omega|p) = \frac{F(\bar{x}(p)|\omega) - \alpha}{F(\bar{x}(p)|\omega) - F(\underline{x}(p)|\omega)}, \tag{2.1}$$

[2]Defining $P(1) = b(x(1))$ would render $P(\cdot)$ discontinuous at $\omega = 1$ when $b(\cdot)$ is discontinuous at $x(1) \in (0, 1)$. The range of $P(\cdot)$ would then include a price, namely $P(1)$, that is not assumed on a nondegenerate interval. This inconvenience is avoided by defining $P(\cdot)$ to be continuous at $\omega = 1$.

where $[\underline{x}(p), \bar{x}(p)) = b^{-1}(p)$. It is straightforward to show that when positive, $\lambda(\omega|p)$ is strictly decreasing in ω. Hence, conditional on an agent's bid being equal to the market-clearing price, he is less likely to end up with the good the higher is the state. Consequently, there is a winner's curse effect associated with rationing; ending up with the good is bad news regarding the state ω. (Of course, rational agents take this into account.)

For convenience, for each $\omega \in [0, 1]$, we extend $\lambda(\omega|\cdot)$ from the range of $P(\cdot)$ to all of $\mathcal{P}$ by defining $\lambda(\omega|p) = 0$ if $p \in \mathcal{P}$ is not in the range of $P(\cdot)$.

If all agents but one employ the bidding function $b(\cdot)$, let $u^\beta(p, x|b(\cdot))$ denote the remaining agent's payoff when his signal is $x \in [0, 1]$, he submits a bid of p, and he is a buyer, and let $u^\sigma(p, x|b(\cdot))$ denote his payoff if he is a seller. To reflect that no single agent can affect the price in this continuum-agent model, the market-clearing price function is the same as that when all agents employ $b(\cdot)$. Hence, $P(\cdot) = b(x(\cdot))$ and we have

$$u^\beta(p, x|b(\cdot)) = \int_{\omega:P(\omega)<p} (v(x, \omega) - P(\omega))h(\omega|x)\,d\omega + \int_{\omega:P(\omega)=p} (v(x, \omega) - p)\lambda(\omega|p)h(\omega|x)\,d\omega$$

and

$$u^\sigma(p, x|b(\cdot)) = \int_{\omega:P(\omega)>p} (P(\omega) - v(x, \omega))h(\omega|x)\,d\omega + \int_{\omega:P(\omega)=p} (p - v(x, \omega))(1 - \lambda(\omega|p))h(\omega|x)\,d\omega,$$

where an integral over the empty set is understood to be zero and where $h(\omega|x) = f(x|\omega)g(\omega)/\int_0^1 f(x|\omega)g(\omega)\,d\omega$ is the conditional density of the state of the good given the agent's signal of x, a notation we shall maintain throughout the proofs.

Consequently,

$$u^\sigma(p, x|b(\cdot)) = u^\beta(p, x|b(\cdot)) + \int_0^1 (P(\omega) - v(x, \omega))h(\omega|x)\,d\omega, \tag{2.2}$$

so that a buyer and seller with the same signal have precisely the same preferences over bids, p. Indeed, (2.2) indicates that a seller can optimize by committing to sell his unit at the market-clearing price and then bid as if he is a buyer. The symmetry between buyers and sellers is a consequence of the fact that when there is a continuum of agents, no single agent can affect the price. This symmetry disappears when there are finitely many agents as will be the case later on.

Because buyers and sellers are symmetric, we may for the sake of convenience and without loss evaluate any agent's payoff (buyer or seller) from bidding p when his signal is x by the common payoff function

$$(2.3)\qquad u(p,x|b(\cdot)) = \int_{\omega:P(\omega)<p} (v(x,\omega)-P(\omega))h(\omega|x)\,d\omega + \int_{\omega:P(\omega)=p} (v(x,\omega)-p)\lambda(\omega|p)h(\omega|x)\,d\omega,$$

where both $P(\cdot)$ and $\lambda(\cdot|\cdot)$ are determined by $b(\cdot)$, the bidding function employed by all other agents.

The nondecreasing bidding function $b(\cdot)$ constitutes a pure symmetric equilibrium of the double auction for $\mathcal{E}(\alpha, v, f, g, \Delta)$ if $b(\cdot)$ is right-continuous on $[0,1]$, continuous at $x=1$, and, given that all other agents employ it, any remaining agent's payoff given by (2.3) is, for each $x\in[0,1]$, maximized when $p=b(x)$. We will henceforth call such a bidding function simply a *double-auction equilibrium for* $\mathcal{E}(\alpha, v, f, g, \Delta)$.

We now investigate the properties of a particularly useful correspondence related to equilibria of $\mathcal{E}(\alpha, v, f, g, \Delta)$. Let $\bar{\mathcal{P}} = \{0, \Delta, 2\Delta, \dots, K\Delta\} = \{p_0, \dots, p_K\}$, where $K\Delta \geq v(1,1) > (K-1)\Delta$. Because no bid above $p_K = K\Delta$ is ever strictly better for any agent than his best bid in $\bar{\mathcal{P}}$, for the purposes of existence it suffices to restrict the agents' bids to $\bar{\mathcal{P}}$.

As in Athey (2001), it will be useful to view nondecreasing step functions as being derived from the points at which they jump. Specifically, a vector of jump points $0 \leq x_1 \leq x_2 \leq \cdots \leq x_K \leq 1$ defines a nondecreasing right-continuous function $b:[0,1]\to\{p_0,\dots,p_K\}$ as follows, where $x_0=0$ and $x_{K+1}=1$:

$$(2.4)\qquad b(x)=p_k, \quad \text{if} \quad x\in[x_k,x_{k+1}) \quad \text{and} \quad b(1)=\lim_{x\uparrow 1} b(x).$$

Note that the definition ensures continuity at $x=1$.

Let X_K denote the nonempty, compact, convex set of nondecreasing vectors in $[0,1]^K$ and let $b_{\mathbf{x}}(\cdot)$ denote the step function defined in (2.4) for $\mathbf{x}\in X_K$.[3] For $\mathbf{x}\in X_K$, suppose that all agents but one employ the bidding function $b_{\mathbf{x}}(\cdot)$. Then $u(p,x|b_{\mathbf{x}}(\cdot))$, given by (2.3), denotes the remaining agent's payoff from bidding $p\in\bar{\mathcal{P}}$ when his signal is x. Because $\mathbf{x}$ represents $b_{\mathbf{x}}(\cdot)$, from now on we shall write $u(p,x|\mathbf{x})$ instead of $u(p,x|b_{\mathbf{x}}(\cdot))$. Consider now the remaining agent's *ex ante* constrained maximization problem

$$(2.5)\qquad \max_{\mathbf{y}\in X_K} \int_0^1 u(b_{\mathbf{y}}(x), x|\mathbf{x}) f(x)\,dx,$$

[3]Note that every nondecreasing right-continuous function that is continuous at $x=1$ and whose range is a subset of $\{p_0,\dots,p_K\}$ is represented by some nondecreasing vector $\mathbf{x}$. In particular, functions that do not assume the price p_k are represented by $\mathbf{x}\in X_K$ such that $x_k=x_{k+1}$.

where $f(x) = \int_0^1 f(x|\omega)g(\omega)\,d\omega$ is the ex ante density over the agent's signal, a notation we maintain throughout the proofs.

This maximization problem is *constrained* because it restricts the agent to nondecreasing bidding functions (i.e., $b_{\mathbf{y}}(\cdot)$) when there is as yet no reason to expect the agent's best reply to be nondecreasing.

Note that $b_{\mathbf{x}}(\cdot)$ would be a double-auction equilibrium if $\mathbf{x}$ were a solution to (2.5) *and* if it could be shown that the remaining agent possesses a nondecreasing best reply when all other agents employ $b_{\mathbf{x}}(\cdot)$.

Let $B(\mathbf{x})$ denote the set of solutions to (2.5).[4] Because the function $\phi(\mathbf{x}, \mathbf{y}) = \int_0^1 u(b_{\mathbf{y}}(x), x|\mathbf{x})f(x)\,dx$ is continuous in $(\mathbf{x}, \mathbf{y})$,[5] $B(\cdot)$ is a nonempty-valued, compact-valued, upper hemicontinuous correspondence from X_K into subsets of itself. However, it need not be convex-valued. Letting $\operatorname{co} B(\mathbf{x})$ denote the convex hull of $B(\mathbf{x})$, it follows from Kakutani's theorem that $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$ for some $\hat{\mathbf{x}} \in X_K$.

We now present two important results concerning the fixed points of $\operatorname{co} B(\cdot)$. A consequence of these results is that whenever $\Delta > 0$ is sufficiently small, every fixed point $\hat{\mathbf{x}}$ of the correspondence $\operatorname{co} B(\cdot)$ is a fixed point of the correspondence $B(\cdot)$, and $b_{\hat{\mathbf{x}}}(\cdot)$ is a double-auction equilibrium for $\mathcal{E}(\alpha, v, f, g, \Delta)$.

In what follows, keep in mind that the grid of prices $\bar{\mathcal{P}}$, the correspondence $B(\cdot)$, and the fixed point $\hat{\mathbf{x}}$ all depend on Δ.

LEMMA 2.1: *There exists $\bar{\eta} > 0$ such that for all K and Δ satisfying $(K-1)\Delta < v(1,1) \leq K\Delta$ and all $\mathbf{x} \in X_K$, if the length of each interval over which $P(\cdot) \equiv b_{\mathbf{x}}(x(\cdot))$ is constant is strictly less than $\bar{\eta}$, then* (i) *$B(\mathbf{x})$ is convex and* (ii) *if all other agents employ $b_{\mathbf{x}}(\cdot)$, then for some $\mathbf{y} \in X_K$ the nondecreasing bidding function $b_{\mathbf{y}}(\cdot)$ maximizes the agent's ex ante* (*and interim*) *payoff among* all *measurable bidding functions*, *nondecreasing or not*.

LEMMA 2.2: *For every $\eta > 0$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, whenever $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$, the length of each interval over which $P(\cdot) \equiv b_{\hat{\mathbf{x}}}(x(\cdot))$ is constant is strictly less than η.*

The proofs of these lemmas will be provided below. We first present an immediate consequence.

PROPOSITION 2.3: *There exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, the following statements are equivalent*:

(a) *$b_{\hat{\mathbf{x}}}(\cdot)$ is a double-auction equilibrium for $\mathcal{E}(\alpha, v, f, g, \Delta)$.*

[4]The correspondence $B(\cdot)$ varies with Δ. Although this dependence is suppressed for simplicity, all of our proofs take it into account.

[5]This follows from the fact that if $(\mathbf{x}^n, \mathbf{y}^n) \to (\mathbf{x}, \mathbf{y})$, then $u(b_{\mathbf{y}^n}(x), x|b_{\mathbf{x}^n}) \to u(b_{\mathbf{y}}(x), x|b_{\mathbf{x}})$ for all values of x but perhaps the finitely many values $x = y_1, \ldots, y_K$ (where $\mathbf{y} = (y_1, \ldots, y_K)$). Applying the dominated convergence theorem yields $\phi(\mathbf{x}^n, \mathbf{y}^n) \to \phi(\mathbf{x}, \mathbf{y})$, as desired.

(b) $\hat{\mathbf{x}} \in B(\hat{\mathbf{x}})$.
(c) $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$.

Consequently, because $\operatorname{co} B(\cdot)$ *always possesses a fixed point, a double-auction equilibrium for* $\mathcal{E}(\alpha, v, f, g, \Delta)$ *exists for all sufficiently small* Δ.

PROOF: Clearly (a) ⇒ (b) ⇒ (c). Hence, it suffices to show that (c) ⇒ (a). If $\Delta > 0$ is sufficiently small, Lemmas 2.2 and 2.1 imply that (i) and (ii) of Lemma 2.1 hold. So, if $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$, then $\hat{\mathbf{x}} \in B(\hat{\mathbf{x}})$ by Lemma 2.1(i). Consequently, $b_{\hat{\mathbf{x}}}(\cdot)$ is a best reply for the remaining agent among all nondecreasing bidding functions when all others employ $b_{\hat{\mathbf{x}}}(\cdot)$. Lemma 2.1(ii) then implies that $b_{\hat{\mathbf{x}}}(\cdot)$ is an equilibrium, as desired. *Q.E.D.*

LEMMA 2.4: *For every* $x \in [0, 1]$, *let* $g(\cdot|x) \geq 0$ *be a density on* $[\underline{\omega}, \bar{\omega}]$. *Further suppose that* $g(\cdot|\cdot)$ *is* C^1 *and satisfies the affiliation inequality on* $[\underline{\omega}, \bar{\omega}] \times [0, 1]$. *Finally, suppose that* $r(x, \omega)$ *is nondecreasing,* C^1, *and satisfies* $r_x(x, \omega) > \varepsilon > 0$ *on* $[\underline{\omega}, \bar{\omega}] \times [0, 1]$. *Let*

$$\phi(x) = \int_{\underline{\omega}}^{\bar{\omega}} r(x, \omega) g(\omega|x)\, d\omega.$$

Then $\phi'(x) > \varepsilon$ *for all* $x \in [0, 1]$.

PROOF: We have

$$\begin{aligned}\phi'(x) &= \int_{\underline{\omega}}^{\bar{\omega}} r_x(x, \omega) g(\omega|x)\, d\omega + \int_{\underline{\omega}}^{\bar{\omega}} r(x, \omega) g_x(\omega|x)\, d\omega \\ &\geq \int_{\underline{\omega}}^{\bar{\omega}} r_x(x, \omega) g(\omega|x)\, d\omega \\ &> \varepsilon,\end{aligned}$$

where the first inequality follows because $r(x, \omega)$ is nondecreasing in ω and $g(\omega|x)$ satisfies the affiliation inequality, and the second follows because $r_x(x, \omega) > \varepsilon$. *Q.E.D.*

PROOF OF LEMMA 2.1: By Theorem 1 in Athey (2001), it suffices to show that there exists $\bar{\eta} > 0$ such that for all K and Δ satisfying $K\Delta \geq v(1, 1) > (K-1)\Delta$ and for all $\mathbf{x} \in X_K$, if the length of each step of $P(\cdot) = b_{\mathbf{x}}(x(\cdot))$ is less than $\bar{\eta}$, then the agent's payoff function $u(p, x|\mathbf{x})$ satisfies the single-crossing property in $(p, x) \in \bar{\mathcal{P}} \times [0, 1]$. That is, for every $\bar{p} > \underline{p}$ in $\bar{\mathcal{P}}$ and every $x \in [0, 1]$, if

$$\begin{aligned}&u(\bar{p}, x|\mathbf{x}) - u(\underline{p}, x|\mathbf{x}) \\ &\quad = \int_{\omega: P(\omega) = \underline{p}} (v(x, \omega) - \underline{p})(1 - \lambda(\omega|\underline{p})) h(\omega|x)\, d\omega\end{aligned}$$

$$
\begin{aligned}
&+\int_{\omega:\underline{p}<P(\omega)<\bar{p}}(v(x,\omega)-P(\omega))h(\omega|x)\,d\omega \\
&+\int_{\omega:P(\omega)=\bar{p}}(v(x,\omega)-\bar{p})\lambda(\omega|\bar{p})h(\omega|x)\,d\omega \\
&\geq 0,
\end{aligned}
$$

then the inequality is maintained when x rises, and if the inequality is strict, the inequality remains strict when x rises.

Noting that the conclusion holds trivially if the interval $\{\omega : \underline{p} \leq P(\omega) \leq \bar{p}\}$ is degenerate, it suffices to show that it holds when the interval $\{\omega : \underline{p} \leq P(\omega) \leq \bar{p}\}$ is nondegenerate.

Let I_0, I_1, and I_2 denote the intervals $P^{-1}(\underline{p})$, $P^{-1}((\underline{p}, \bar{p}))$, and $P^{-1}(\bar{p})$, respectively. At least one of the I_k is nondegenerate because $I_0 \cup I_1 \cup I_2 = \{\omega : \underline{p} \leq P(\omega) \leq \bar{p}\}$ is nondegenerate. For $x \in [0, 1]$, define

$$
\begin{aligned}
d_0(x) &= \int_{I_0}(1-\lambda(\omega|\underline{p}))h(\omega|x)\,d\omega, \\
d_1(x) &= \int_{I_1} h(\omega|x)\,d\omega, \\
d_2(x) &= \int_{I_2} \lambda(\omega|\bar{p})h(\omega|x)\,d\omega.
\end{aligned} \tag{2.6}
$$

Also for $x \in [0, 1]$, define the density $h_k(\cdot|x)$ on I_k, for $k = 0, 1, 2$, as

$$
\begin{aligned}
h_0(\omega|x) &= (1-\lambda(\omega|\underline{p}))h(\omega|x)/d_0(x), \quad \text{if} \quad \omega \in I_0, \\
h_1(\omega|x) &= h(\omega|x)/d_1(x), \quad \text{if} \quad \omega \in I_1, \\
h_2(\omega|x) &= \lambda(\omega|\bar{p}h)(\omega|x)/d_2(x), \quad \text{if} \quad \omega \in I_2,
\end{aligned}
$$

where $h_k(\omega|x)$ is defined to be zero if the denominator $d_k(x)$ that appears in its definition is zero.

Letting $\phi(x) = u(\bar{p}, x|\mathbf{x}) - u(\underline{p}, x|\mathbf{x})$, we then have

$$
\begin{aligned}
\phi(x) = d_0(x)&\int_{I_0}(v(x,\omega)-\underline{p})h_0(\omega|x)\,d\omega \\
&+ d_1(x)\int_{I_1}(v(x,\omega)-P(\omega))h_1(\omega|x)\,d\omega \\
&+ d_2(x)\int_{I_2}(v(x,\omega)-\bar{p})h_2(\omega|x)\,d\omega.
\end{aligned}
$$

Now, because $I_0 \cup I_1 \cup I_2 = \{\omega : \underline{p} \leq P(\omega) \leq \bar{p}\}$ is nondegenerate, $d_0(x) + d_1(x) + d_2(x) > 0$ for all $x \in [0, 1]$. For $x \in [0, 1]$, define, for $k = 0, 1, 2$,

$$a_k(x) = d_k(x)/(d_0(x) + d_1(x) + d_2(x)),$$

$$c_0(x) = \int_{I_0} (v(x, \omega) - \underline{p})h_0(\omega|x)\, d\omega,$$

$$c_1(x) = \int_{I_1} (v(x, \omega) - P(\omega))h_1(\omega|x)\, d\omega,$$

$$c_2(x) = \int_{I_2} (v(x, \omega) - \bar{p})h_2(\omega|x)\, d\omega$$

and define $\gamma(x) = \phi(x)/(d_0(x) + d_1(x) + d_2(x))$ for $x \in [0, 1]$. Then

$$\gamma(x) = a_0(x)c_0(x) + a_1(x)c_1(x) + a_2(x)c_2(x). \tag{2.7}$$

Note that $\operatorname{sgn} \phi(x) = \operatorname{sgn} \gamma(x)$, because $d_0(x) + d_1(x) + d_2(x) > 0$ on $[0, 1]$. Hence, it suffices to show that there exists $\bar{\eta} > 0$ such that for all $\mathbf{x} \in X_K$, if the length of every step of $P(\cdot) = b_{\mathbf{x}}(x(\cdot))$ is less than $\bar{\eta}$, then $\gamma'(x) > 0$ for all $x \in [0, 1]$ and for all pairs of prices $\underline{p} < \bar{p}$ in $\bar{\mathcal{P}}$ such that $[\underline{p}, \bar{p}]$ contains a price in the range of $P(\cdot)$.

Now

$$\gamma'(x) = a_0(x)c_0'(x) + a_1(x)c_1'(x) + a_2(x)c_2'(x) + a_0'(x)c_0(x) + a_1'(x)c_1(x) + a_2'(x)c_2(x).$$

Moreover, because $v_x(x, \omega) > 0$ and continuous on $[0, 1]^2$, there exists $\varepsilon > 0$ such that $v_x(x, \omega) > \varepsilon$ on $[0, 1]^2$. Consequently, Lemma 2.4 implies that $c_k'(x) > \varepsilon$ for all x and for $k = 0, 1, 2$. Because the $a_k(x)$ are nonnegative and sum to 1, this implies

$$\gamma'(x) > \varepsilon + a_0'(x)c_0(x) + a_1'(x)c_1(x) + a_2'(x)c_2(x).$$

Because $c_k(x)$ is bounded on $[0, 1]$, it suffices to show that the $a_k'(x)$ are sufficiently close to zero uniformly in x when the step widths of $P(\cdot)$ are sufficiently small. More precisely, it suffices to show that for all sequences $K_r \to_r \infty$ and $\Delta_r \to_r 0$ such that $K_r\Delta_r \geq v(1, 1) > (K_r - 1)\Delta_r$, for every sequence of price functions $P_r : [0, 1] \to \bar{\mathcal{P}}_r = \{0, \Delta_r, \ldots, K_r\Delta_r\}$ whose step widths converge uniformly to zero, and for all sequences of prices $\underline{p}_r < \bar{p}_r$ in $\bar{\mathcal{P}}_r$ such that $[\underline{p}_r, \bar{p}_r]$ contains a price in the range of $P_r(\cdot)$, the corresponding sequences $a_{0,r}'(x)$, $a_{1,r}'(x)$, and $a_{2,r}'(x)$ converge to zero uniformly in $x \in [0, 1]$. Given such sequences K_r, Δ_r, $P_r(\cdot)$, $\underline{p}_r$, and $\bar{p}_r$, we now derive the desired conclusion. However, to simplify the notation, we suppress the index r.

As the width of the steps of $P(\cdot)$ converges uniformly to zero, the length of the intervals I_0 and I_2 converge to zero because each of these corresponds to a length of one of the steps of $P(\cdot)$. However, the length of I_1 need not converge to zero because I_1 is potentially the sum of the lengths of many steps of $P(\cdot)$ and the number of these steps might be unbounded along the sequence. Of course, the length of I_1 is bounded.

Of the three derivatives, $a_0'(x)$, $a_1'(x)$, and $a_2'(x)$, we will only treat one in detail; the other two are similar. Let us consider $a_0'(x)$. Direct computation yields

$$a_0'(x) = \left[\frac{d_0'(x)}{d_0(x)} - \frac{d_1'(x)}{d_1(x)}\right]a_0(x)a_1(x) + \left[\frac{d_0'(x)}{d_0(x)} - \frac{d_2'(x)}{d_2(x)}\right]a_0(x)a_2(x), \tag{2.8}$$

where for $k = 0, 1, 2$, $d_k'(x)/d_k(x)$ is understood to be zero if I_k is degenerate, the latter being the only instance in which $d_k(x) = a_k(x) = 0$.

If I_0 is degenerate at any point in the sequence, then $a_0(\cdot)$ and $a_0'(\cdot)$ are identically zero at that point in the sequence. So, without loss, we may assume that along the sequence, I_0 is always nondegenerate.

By (2.8), it suffices to show that

$$\left[\frac{d_0'(x)}{d_0(x)} - \frac{d_1'(x)}{d_1(x)}\right]a_0(x)a_1(x) \tag{2.9}$$

and

$$\left[\frac{d_0'(x)}{d_0(x)} - \frac{d_2'(x)}{d_2(x)}\right]a_0(x)a_2(x) \tag{2.10}$$

each converge uniformly to zero.

Now, along the sequence, the intervals I_1 and I_2 can each be either degenerate or nondegenerate. Any subsequence along which both are degenerate is such that both (2.9) and (2.10) are zero for all x along that subsequence (recall the convention established in (2.8)). We consider the remaining cases one at a time.

CASE I: Consider any subsequence along which I_1 is degenerate and I_2 is nondegenerate. Because I_1 is degenerate, (2.9) is zero for all x. The definition of $d_0(x)$ yields

$$\begin{aligned} \frac{d_0'(x)}{d_0(x)} &= \frac{\int_{I_0}(1-\lambda(\omega|\underline{p}))h_x(\omega|x)\,d\omega}{\int_{I_0}(1-\lambda(\omega|\underline{p}))h(\omega|x)\,d\omega} \\ &= \int_{I_0}\frac{h_x(\omega|x)}{h(\omega|x)}\left(\frac{(1-\lambda(\omega|\underline{p}))h(\omega|x)}{\int_{I_0}(1-\lambda(\omega|\underline{p}))h(\omega|x)d\omega}\right)d\omega \\ &\to \frac{h_x(\omega_0|x)}{h(\omega_0|x)}, \end{aligned} \tag{2.11}$$

where ω_0 is the common limit of the upper and lower endpoints of the sequence of intervals $\{I_0\}$, whose lengths converge to zero, and where the limit in the display follows because the expression on the right-hand side of the second equality is an average of $h_x(\omega|x)/h(\omega|x)$ over $\omega \in I_0$, and both $h_x(\cdot|x)$ and $h(\cdot|x)$ are continuous and $h(\cdot|x) > 0$. Moreover, the convergence is uniform in x because $h_x(\omega|x)$ and $h(\omega|x)$ are jointly continuous in ω and x.[6]

Similarly,

$$\frac{d_2'(x)}{d_2(x)} \to \frac{h_x(\omega_2|x)}{h(\omega_2|x)} \quad \text{uniformly in } x,$$

where ω_2 is the common limit of the upper and lower endpoints of the subsequence of nondegenerate intervals $\{I_2\}$, whose lengths converge to zero.

Now, because I_1 is degenerate along the subsequence under consideration, the intervals I_0 and I_2 are adjacent and so their endpoints must converge to a common point. That is, $\omega_0 = \omega_2$. We conclude that (2.10) converges to zero uniformly in x, because $a_0(x)$ and $a_2(x)$ are bounded between 0 and 1. Hence, $a_0'(x)$ converges to zero uniformly in x.

Case II: Next, consider any subsequence along which I_2 is degenerate and I_1 is nondegenerate. Because I_2 is degenerate, (2.10) is zero for all x along the subsequence. If the length of I_1 converges to zero along any further subsequence, then as above we may conclude that

$$\frac{d_1'(x)}{d_1(x)} \to \frac{h_x(\omega_1|x)}{h(\omega_1|x)} \quad \text{uniformly in } x, \tag{2.12}$$

where ω_1 is the common limit of the upper and lower endpoints of the further subsequence of intervals $\{I_1\}$. However, because I_0 and I_1 are adjacent intervals, we must then have $\omega_0 = \omega_1$ so that (2.9) converges to zero uniformly in x. Hence, $a_0'(x)$ converges to zero uniformly in x.

If the length of I_1 is bounded away from zero along some other further subsequence, then $d_0(x)/d_1(x)$ converges to zero uniformly in x, so that $a_0(x)$ converges to zero uniformly in x. Because $d_1'(x)/d_1(x)$ is bounded on $[0, 1]$ along the subsequence, (2.9) converges to zero uniformly in x.

Case III: Consider any subsequence along which both I_1 and I_2 are nondegenerate. If the length of I_1 converges to zero along any further subsequence, then once again (2.12) holds, and again $\omega_0 = \omega_1$ so that (2.9) converges to zero uniformly in x. However, I_1 and I_2 are also adjacent intervals, so that, similarly, $\omega_0 = \omega_1 = \omega_2$ and so (2.10) converges to zero uniformly in x as well. Hence, $a_0'(x)$ converges to zero uniformly in x.

[6]Note that this limit result takes into account the fact that the function $\lambda(\cdot|\underline{p})$ varies with I_0 and $\underline{p}$.

If the length of I_1 is bounded away from zero along some other further subsequence, then both $d_0(x)/d_1(x)$ and $d_2(x)/d_1(x)$ converge to zero uniformly in x, so that both $a_0(x)$ and $a_2(x)$ converge to zero uniformly in x. Because $d_k'(x)/d_k(x)$, being an average over ω of $h_x(\omega|x)/h(\omega|x)$ (see, e.g., (2.11)), is bounded on [0, 1] along the subsequence, (2.9) and (2.10) each converge to zero uniformly in x, and hence so does $a_0'(x)$.

Having considered all possible cases, we conclude that $a_0'(x)$ converges to zero uniformly in x. A similar argument establishes the required uniform convergence to zero of both $a_1'(x)$ and $a_2'(x)$. *Q.E.D.*

Note that the preceding proof did not make essential use of the exact functional form of the rationing probability function $\lambda(\omega|p)$. We have therefore actually proven the following result, which we record here for future reference.

COROLLARY 2.5: *There exists* $\bar{\eta} > 0$ *such that for all* K *and* Δ, *all* $\bar{\mathcal{P}} = \{0, \Delta, \dots, K\Delta\}$, *all right-continuous nondecreasing functions* $P : [0, 1] \to \bar{\mathcal{P}}$ *continuous at* 1, *and all measurable* $\gamma : [0, 1] \times \bar{\mathcal{P}} \to [0, 1]$ *such that* $\gamma(\omega|p) \in (0, 1)$ *whenever* $P(\omega) = p$, *if the width of each step of* $P(\cdot)$ *is strictly less than* $\bar{\eta}$ *and*

$$r(p, x) = \int_{\omega: P(\omega) < p} (v(x, \omega) - P(\omega))h(\omega|x)\, d\omega + \int_{\omega: P(\omega) = p} (v(x, \omega) - p)\gamma(\omega|p)h(\omega|x)\, d\omega,$$

then, for all $\bar{p} > \underline{p}$ *in* $\bar{\mathcal{P}}$ *such that* $[\underline{p}, \bar{p}]$ *contains a price in the range of* $P(\cdot)$, *there is a strictly positive* C^1 *function* $d(\cdot)$ *on* [0, 1] *such that*

$$\frac{d}{dx}\frac{r(\bar{p}, x) - r(\underline{p}, x)}{d(x)} \geq \bar{\eta} > 0 \tag{2.13}$$

for all $x \in [0, 1]$.[7]

PROOF OF LEMMA 2.2: Suppose, by way of contradiction, that the lemma is false. Then there is a sequence of finite price grids $\bar{\mathcal{P}}^{\Delta}$ that become dense in $[0, v(1, 1)]$ as $\Delta \to 0$ and each of whose highest price lies between $v(1, 1)$ and $v(1, 1) + 1$, and there is a corresponding sequence of fixed points $\mathbf{x}^{\Delta}$ of $\operatorname{co} B(\cdot)$ such that for some $\varepsilon > 0$ and for every $\Delta > 0$, the width of some step of $P^{\Delta}(\cdot) = b_{\mathbf{x}^{\Delta}}(x(\cdot))$ is of length at least ε.[8] That is, for each Δ, there is an interval of states $[\underline{\omega}^{\Delta}, \bar{\omega}^{\Delta})$ on which $P^{\Delta}(\cdot)$ assumes the value p^{Δ}, say, and such

[7]Let $d(\cdot) = d_0(\cdot) + d_1(\cdot) + d_2(\cdot)$, where the d_k are as in the proof of Lemma 2.1.

[8]The number of prices in $\bar{\mathcal{P}}^{\Delta}$, and so the dimension of $\mathbf{x}^{\Delta}$, also depends on Δ and will typically increase without bound as Δ tends to zero.

that $\bar{\omega}^\Delta - \underline{\omega}^\Delta \geq \varepsilon$ for all Δ. Because $p^\Delta \in [0, v(1,1)+1]$, we may assume that $p^\Delta \to p$ as $\Delta \to 0$.

Since the length of the interval of states on which $P^\Delta(\cdot) = b_{\mathbf{x}^\Delta}(x(\cdot))$ is p^Δ is bounded away from zero and $x(\cdot)$ is continuous and strictly increasing, the length of the interval of signals on which $b_{\mathbf{x}^\Delta}(\cdot)$ is p^Δ must also be bounded away from zero. So, for every Δ, we may suppose that this interval of signals has length at least $\eta > 0$.

Now, $\mathbf{x}^\Delta \in \operatorname{co} B(\mathbf{x}^\Delta)$ implies that $\mathbf{x}^\Delta$ is a convex combination of finitely many vectors $\mathbf{y}^\Delta, \ldots, \mathbf{z}^\Delta$ in $B(\mathbf{x}^\Delta)$. Because the length of the interval on which $b_{\mathbf{x}^\Delta}(\cdot)$ is p^Δ is at least η, the same must be true for at least one of $b_{\mathbf{y}^\Delta}(\cdot), \ldots, b_{\mathbf{z}^\Delta}(\cdot)$. Hence, for every Δ there exists $\mathbf{y}^\Delta \in B(\mathbf{x}^\Delta)$ such that the length of the interval, $[\underline{y}^\Delta, \bar{y}^\Delta)$ say, on which $b_{\mathbf{y}^\Delta}(\cdot)$ is p^Δ is at least η.

Because $\mathbf{y}^\Delta \in B(\mathbf{x}^\Delta)$, $b_{\mathbf{y}^\Delta}(\cdot)$ maximizes the agent's ex ante payoff when all others employ $b_{\mathbf{x}^\Delta}(\cdot)$ and the agent is restricted to nondecreasing strategies. Hence, the agent's interim payoff when his signal is $\underline{y}^\Delta$ must be at least as high when he bids p^Δ as it would be were he to bid the next price in the grid below p^Δ (assuming, for the moment, that $p^\Delta > 0$ so that such a price exists). Otherwise, because the agent's payoff is continuous in his signal, he can increase his ex ante payoff, while still satisfying the nondecreasing strategy constraint, by reducing his bid just below p^Δ for a small interval of signals around $\underline{y}^\Delta$. So, if p_-^Δ denotes the price in $\bar{\mathcal{P}}^\Delta$ just below p^Δ, we must have

$$\text{(2.14)} \qquad \int_{\underline{\underline{\omega}}^\Delta}^{\underline{\omega}^\Delta} (v(\underline{y}^\Delta, \omega) - p_-^\Delta)(1 - \lambda(\omega | p_-^\Delta)) h(\omega | \underline{y}^\Delta)\, d\omega + \int_{\underline{\omega}^\Delta}^{\bar{\omega}^\Delta} (v(\underline{y}^\Delta, \omega) - p^\Delta) \lambda(\omega | p^\Delta) h(\omega | \underline{y}^\Delta)\, d\omega \geq 0,$$

where $[\underline{\underline{\omega}}^\Delta, \underline{\omega}^\Delta)$ is the (possibly empty) interval on which $P^\Delta(\cdot)$ assumes the value p_-^Δ and where $[\underline{\omega}^\Delta, \bar{\omega}^\Delta)$ is the interval on which $P^\Delta(\cdot)$ assumes the value p^Δ. We have so far assumed that $p^\Delta > 0$. If $p^\Delta = 0$, define $[\underline{\underline{\omega}}^\Delta, \underline{\omega}^\Delta)$ to be empty so that (2.14) clearly holds. Hence, with this convention, (2.14) holds for all $p^\Delta \geq 0$.

Similarly, letting p_+^Δ denote the price in $\bar{\mathcal{P}}^\Delta$ just above p^Δ, the difference in the agent's payoff from bidding p_+^Δ versus p^Δ when his signal is $\bar{y}^\Delta$ must be nonpositive. Hence,

$$\text{(2.15)} \qquad \int_{\underline{\omega}^\Delta}^{\bar{\omega}^\Delta} (v(\bar{y}^\Delta, \omega) - p^\Delta)(1 - \lambda(\omega | p^\Delta)) h(\omega | \bar{y}^\Delta)\, d\omega + \int_{\bar{\omega}^\Delta}^{\bar{\bar{\omega}}^\Delta} (v(\bar{y}^\Delta, \omega) - p_+^\Delta) \lambda(\omega | p_+^\Delta) h(\omega | \bar{y}^\Delta)\, d\omega \leq 0,$$

where $[\bar{\omega}^{\Delta}, \overline{\overline{\omega}}^{\Delta})$ is the (possibly empty) interval on which $P^{\Delta}(\cdot)$ assumes the value p_{+}^{Δ}. If p^{Δ} is the highest price in $\bar{\mathcal{P}}^{\Delta}$, then define $[\bar{\omega}^{\Delta}, \overline{\overline{\omega}}^{\Delta})$ to be empty, so that (2.15) holds because the highest price in the grid is at least $v(1,1)$.

Let us now consider the limits of (2.14) and (2.15) as $\Delta \to 0$. Without loss, we may assume that $\underline{y}^{\Delta} \to \underline{y}$, $\bar{y}^{\Delta} \to \bar{y}$, $\underline{\underline{\omega}}^{\Delta} \to \underline{\underline{\omega}}$, $\underline{\omega}^{\Delta} \to \underline{\omega}$, $\bar{\omega}^{\Delta} \to \bar{\omega}$, and $\overline{\overline{\omega}}^{\Delta} \to \overline{\overline{\omega}}$. Note that because $\bar{\omega}^{\Delta} - \underline{\omega}^{\Delta} \geq \varepsilon$ and $\bar{y}^{\Delta} - \underline{y}^{\Delta} \geq \eta$ for all Δ, $[\underline{\omega}, \bar{\omega})$ and $[\underline{y}, \bar{y})$ are nondegenerate. On the other hand, the length of either of the intervals $[\underline{\underline{\omega}}, \underline{\omega})$ and $[\bar{\omega}, \overline{\overline{\omega}})$ might be zero.

Recall that $p^{\Delta} \to p$. Hence, p_{-}^{Δ} and p_{+}^{Δ} also converge to p. Using the definition of $\lambda(\cdot|\cdot)$, one can directly compute the limits of $\lambda(\omega|p_{-}^{\Delta})$, $\lambda(\omega|p^{\Delta})$, and $\lambda(\omega|p_{+}^{\Delta})$, which exist, respectively, whenever the intervals $[\underline{\underline{\omega}}, \underline{\omega})$, $[\underline{\omega}, \bar{\omega})$, and $[\bar{\omega}, \overline{\overline{\omega}})$ are nonempty. Hence, when all three intervals are nonempty, there are nonincreasing functions $\gamma_{-}(\cdot)$, $\gamma(\cdot)$, and $\gamma_{+}(\cdot)$, each taking values in $[0,1]$, such that $\lambda(\omega|p_{-}^{\Delta}) \to \gamma_{-}(\omega)$, $\lambda(\omega|p^{\Delta}) \to \gamma(\omega)$, and $\lambda(\omega|p_{+}^{\Delta}) \to \gamma_{+}(\omega)$ for all $\omega \in [\underline{\underline{\omega}}, \underline{\omega}]$, $[\underline{\omega}, \bar{\omega}]$, and $[\bar{\omega}, \overline{\overline{\omega}}]$, respectively. (If an interval is empty, its corresponding limit function can be defined arbitrarily because the limit of the corresponding integral is zero.) Moreover, because $[\underline{\omega}, \bar{\omega})$ and $[\underline{y}, \bar{y})$ are nondegenerate, the strict monotone likelihood ratio property (MLRP) implies that $\gamma(\cdot)$ is strictly decreasing on $[\underline{\omega}, \bar{\omega}]$ unless $\underline{y} = 0$ and $\bar{y} = 1$, in which case γ is constant and equal to $1 - \alpha$; either way, $\gamma(\cdot)$ is not almost everywhere equal to 1 on $[\underline{\omega}, \bar{\omega}]$. So, because $v(x, \omega)$ and $h(\omega|x)$ are continuous in (ω, x), Lebesgue's dominated convergence theorem implies that the limits of (2.14) and (2.15) as $\Delta \to 0$ are, respectively,

$$\text{(2.16)} \qquad \int_{\underline{\underline{\omega}}}^{\underline{\omega}} (v(\underline{y}, \omega) - p)(1 - \gamma_{-}(\omega)) h(\omega|\underline{y})\, d\omega + \int_{\underline{\omega}}^{\bar{\omega}} (v(\underline{y}, \omega) - p)\gamma(\omega) h(\omega|\underline{y})\, d\omega \geq 0$$

and

$$\text{(2.17)} \qquad \int_{\underline{\omega}}^{\bar{\omega}} (v(\bar{y}, \omega) - p)(1 - \gamma(\omega)) h(\omega|\bar{y})\, d\omega + \int_{\bar{\omega}}^{\overline{\overline{\omega}}} (v(\bar{y}, \omega) - p)\gamma_{+}(\omega) h(\omega|\bar{y})\, d\omega \leq 0.$$

Now, either $v(\underline{y}, \underline{\omega}) - p < 0$ or $v(\underline{y}, \underline{\omega}) - p \geq 0$. In the former case, because $v(x, \omega)$ is nondecreasing, the first integral in (2.16) is nonpositive and so the second is nonnegative. In the latter case, $v(x, \omega)$ being nondecreasing directly implies that the second integral in (2.16) is nonnegative. Hence, in either case, the second integral in (2.16) is nonnegative. A similar argument establishes that the first integral in (2.17) is nonpositive. However, because $\bar{y} > \underline{y}$ and because $\gamma(\cdot)$ is not almost everywhere equal to 1,

$$\int_{\underline{\omega}}^{\bar{\omega}} (v(\bar{y}, \omega) - p)\left[\frac{(1-\gamma(\omega))h(\omega|\bar{y})}{\int_{\underline{\omega}}^{\bar{\omega}}(1-\gamma(\omega))h(\omega|\bar{y})\,d\omega}\right] d\omega$$

$$\geq \int_{\underline{\omega}}^{\bar{\omega}} (v(\bar{y}, \omega) - p)\left\{\frac{\gamma(\omega)h(\omega|\bar{y})}{\int_{\underline{\omega}}^{\bar{\omega}}\gamma(\omega)h(\omega|\bar{y})\,d\omega}\right\} d\omega$$

$$> \int_{\underline{\omega}}^{\bar{\omega}} (v(\underline{y}, \omega) - p)\left\{\frac{\gamma(\omega)h(\omega|\underline{y})}{\int_{\underline{\omega}}^{\bar{\omega}}\gamma(\omega)h(\omega|\underline{y})\,d\omega}\right\} d\omega,$$

where the first inequality follows because $v(x, \omega)$ is nondecreasing and the density in square brackets first-order stochastically dominates the density in curly brackets,[9] and the second inequality follows from Lemma 2.4. Hence, it cannot be the case that the second integral in (2.16) is nonnegative and the first integral in (2.17) is nonpositive. This contradiction completes the proof. *Q.E.D.*

Suppose that $b:[0,1] \to \bar{\mathcal{P}}$ is a double-auction equilibrium for $\mathcal{E}(\alpha, v, f, g, \Delta)$ and that the range of $P(\cdot) \equiv b(x(\cdot))$ is $\pi_1 < \cdots < \pi_{L-1}$.[10] For $l = 1, \ldots, L-1$, let $[x_l, x_{l+1})$ denote the (nonempty) interval on which $b(\cdot)$ is π_l. Then

$$x_1 \leq x(0) < x_2 \quad \text{and} \quad x_{L-1} < x(1) \leq x_L.$$

According to Lemma 2.2, when the grid of prices is fine enough (i.e., when Δ is sufficiently small), $P(\cdot)$ has uniformly narrow steps. Consequently, because on $[0, 1]$, $x'(\omega)$ is positive and continuous (and so bounded away from zero) and $P(\cdot) = b(x(\cdot))$ for every $\varepsilon > 0$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, $x_{l+1} - x_l < \varepsilon$ whenever both x_l and x_{l+1} are in $[x(0), x(1)]$. Our next result, maintaining the notation here, sharpens this to show also that $x_2 - x_1 < \varepsilon$ and $x_L - x_{L-1} < \varepsilon$. The dependence of each x_l and π_l, and of L on Δ is suppressed throughout.

LEMMA 2.6: *For every $\varepsilon > 0$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$ and for every double-auction equilibrium, $|x_1 - x(0)| < \varepsilon$, $|x_L - x(1)| < \varepsilon$, and $0 < x_l - x_{l-1} < \varepsilon$ for all $l = 2, \ldots, L$. Hence, $L \to \infty$ as $\varepsilon \to 0$.*

[9]In particular, for every pair of states $\bar{\omega} > \underline{\omega}$, the relative likelihood of $\bar{\omega}$ versus $\underline{\omega}$ is higher under the density in square brackets because $\lambda(\omega)$ is a nonincreasing function of ω.

[10]Hence, each $\pi_l = k_l\Delta$ for some $k_l \in \{0, 1, \ldots, K\}$.

PROOF: In view of the discussion preceding the statement of the lemma and because Lemma 2.2 implies that $x_2 \to x(0)$ and $x_{L-1} \to x(1)$ as $\Delta \to 0$, it suffices to show that $x_1 \to x(0)$ and $x_L \to x(1)$ as $\Delta \to 0$. We provide the argument only for x_1, because the other argument is similar. Assume, by way of contradiction, that x_1 does not converge to $x(0)$ as $\Delta \to 0$. Then, because $x_1 \le x(0)$, we may assume that $x_1 \to x_1^* < x(0)$.

Because an agent can always bid zero, his payoff must be nonnegative. In particular, this must be the case when his signal is x_1 and he submits his equilibrium bid of π_1. That is,

$$\int_{\omega:x(\omega)\in[x_1,x_2)} (v(x_1,\omega)-\pi_1)\lambda(\omega|\pi_1)h(\omega|x_1)\,d\omega \ge 0. \tag{2.18}$$

Now, because for Δ small enough, $x_1 < x(0) < x_2$, the set $\{\omega : x(\omega) \in [x_1, x_2)\} = [0, \omega(x_2))$ is nondegenerate.[11] Consequently, we may divide (2.18) by the positive quantity $\int_0^{\omega(x_2)} \lambda(\omega|\pi_1)h(\omega|x_1)\,d\omega$, yielding

$$\int_0^{\omega(x_2)} v(x_1,\omega)\frac{\lambda(\omega|\pi_1)h(\omega|x_1)}{\int_0^{\omega(x_2)}\lambda(\omega|\pi_1)h(\omega|x_1)\,d\omega}\,d\omega \ge \pi_1. \tag{2.19}$$

Note that $x_2 \to x(0)$ implies $\omega(x_2) \to 0$. Consequently, because $v(x,\omega)$ is continuous, taking the limit of (2.19) as $\Delta \to 0$ gives, assuming without loss that $\pi_1 \to \pi^*$ (each π_l, a member of the changing price grid, depends on Δ),

$$v(x_1^*, 0) \ge \pi^*. \tag{2.20}$$

Consider now $\bar{x} \in (x_1^*, x(0))$. Because $v_x > 0$, $v(\bar{x}, 0) > \pi^* + \eta$ for some $\eta > 0$. Consequently, for Δ small enough, it would be strictly better for an agent with signal $x \in (\bar{x}, x(0))$ to bid $\pi' \in (\pi_1, \pi^* + \eta)$ than to bid π_1, where π' is the price just above π_1 in the grid. (Recall that the grid becomes arbitrarily fine as Δ tends to zero.) This is because whether the price of the good is π' or π_1, such an agent strictly prefers to end up with it. Bidding π' guarantees that the agent ends up with the good if the price is π_1 and gives the agent a positive probability of ending up with the good if the price is π'. Bidding π_1 gives no chance of ending up with the good if the price is π' and only gives a probability less than 1 of ending up with it when the price is π_1. Consequently, for Δ small enough, all agents with signals $x \in (\bar{x}, x(0))$ are not optimizing by bidding π_1. (This argument is valid whether or not $P(\cdot)$ assumes the price π'.) This contradiction completes the argument. *Q.E.D.*

An implication of Lemma 2.6 is that if the grid size is small enough, then to each double-auction equilibrium $b(\cdot)$ there are prices $0 < \pi_1 < \cdots < \pi_{L-1} <$

[11] Recall that $\omega(x)$ is the state in which the αth percentile is closest to x. See RP Section 3.

$K\Delta$ and signals $0 < x_1 < \cdots < x_L < 1$, where $L \geq 2$, such that $x_1 \leq x(0) < x_2$ and $x_{L-1} < x(1) \leq x_L$, and for each $l = 1, \ldots, L-1$,

$$b(x) = \pi_l \quad \text{for all } x \in [x_l, x_{l+1}), \tag{2.21}$$

and where the range of $P(\cdot) \equiv b(x(\cdot))$ is $\{\pi_1, \ldots, \pi_{L-1}\}$.

The reader should keep in mind that $(x_1, \ldots, x_L)$ is not in general the jump-point representation of $b(\cdot)$ as defined in (2.4). There are two reasons for this. First, as we have seen, when Δ is sufficiently small, $b(\cdot)$ takes on values strictly below π_1 and strictly above π_{L-1}, and so there are additional jump points. Second, the π_l need not be consecutive grid prices because $b(\cdot)$, and hence $P(\cdot)$, might jump from one grid price to another, while skipping over several in between. In each case, the dimension of the jump-point vector that represents $b(\cdot)$ as defined in (2.4) will be strictly greater than $L-1$.

Nonetheless, the π_l and x_l are sufficient for describing the equilibrium $b(\cdot)$ because all bids below π_1 (resp., above π_{L-1}) are equivalent to one another because they ensure that the bidder does not (resp., does) end up with the good, and switching one such bid for another has no effect on the equilibrium outcome and the resulting bid function remains in equilibrium. Furthermore, the missing jump-point dimensions are redundant because each missing entry, for prices between π_1 and π_{L-1}, would be equal to one of the x_l.

Now, in equilibrium, each agent's bid, for $l = 2, \ldots, L$, jumps from π_{l-1} to π_l when his signal is x_l and so the agent must be indifferent between the two bids. Also, when x_1 is strictly positive, an agent with signal x_1 must be indifferent between bidding π_1 and any price in the grid strictly below it because all such lower prices leave the agent without a unit. Similarly, when x_L is strictly less than 1, an agent with signal x_L must be indifferent between bidding π_{L-1} and any price in the grid strictly above it.

Consequently, for all Δ small enough, because $x_1 > 0$ and $x_L < 1$,

$$\begin{aligned} &\int_{\omega : x(\omega) \in [x_{l-1}, x_l)} (v(x_l, \omega) - \pi_{l-1})(1 - \lambda(\omega|\pi_{l-1}))h(\omega|x_l)\,d\omega \\ &\qquad + \int_{\omega : x(\omega) \in [x_l, x_{l+1})} (v(x_l, \omega) - \pi_l)\lambda(\omega|\pi_l)h(\omega|x_l)\,d\omega = 0, \end{aligned} \tag{2.22}$$

must hold for all $l = 1, 2, \ldots, L$, where we define $x_0 = \pi_0 = 0$, $x_{L+1} = 1$, and $\pi_L = K\Delta$. Note then that when $l = 1$, the first integral (which is the only place where π_0 appears) is zero regardless of the value of π_0, because it is integrated over the set $\{\omega : x(\omega) \in [0, x_1)\}$, which is empty because $x_1 \leq x(0)$. Similarly, when $l = L$, the second integral (which is the only place where π_L appears) is zero regardless of the value of π_L. Consequently, the definitions of π_0 and π_L are rather arbitrary.

By the definition of λ, for every ω such that $x(\omega) \in [x_l, x_{l+1})$ and for every $l = 0, 1, 2, \ldots, L$,

$$\lambda(\omega|\pi_l) = \frac{F(x_{l+1}|\omega) - \alpha}{F(x_{l+1}|\omega) - F(x_l|\omega)}.$$

So, if we define

$$\bar{\lambda}(\omega|x_l, x_{l+1}) = \frac{F(x_{l+1}|\omega) - \alpha}{F(x_{l+1}|\omega) - F(x_l|\omega)}$$

and substitute into (2.22), we obtain

$$\begin{aligned} &\int_{\omega: x(\omega) \in [x_{l-1}, x_l)} (v(x_l, \omega) - \pi_{l-1})(1 - \bar{\lambda}(\omega|x_l, x_{l+1}))h(\omega|x_l)\, d\omega \\ &\quad + \int_{\omega: x(\omega) \in [x_l, x_{l+1})} (v(x_l, \omega) - \pi_l)\bar{\lambda}(\omega|x_l, x_{l+1})h(\omega|x_l)\, d\omega = 0 \end{aligned} \tag{2.23}$$

for $l = 1, 2, \ldots, L$.

We next obtain a normalized version of the system of L equations in (2.23) by dividing the lth equation by

$$\begin{aligned} &\int_{\omega: x(\omega) \in [x_{l-1}, x_l)} (1 - \bar{\lambda}(\omega|x_{l-1}, x_l))h(\omega|x_l)\, d\omega \\ &\quad + \int_{\omega: x(\omega) \in [x_l, x_{l+1})} \bar{\lambda}(\omega|x_l, x_{l+1})h(\omega|x_l)\, d\omega. \end{aligned} \tag{2.24}$$

Note that this quantity is nonzero because $0 = x_0 < x_1 < \cdots < x_L < x_{L+1} = 1$ by (2.21).

The resulting equations for $l = 1, \ldots, L$ take the form

$$\begin{aligned} &(1 - \beta(x_{l-1}, x_l, x_{l+1})) \\ &\quad \times \left(\int_{\omega: x(\omega) \in [x_{l-1}, x_l)} v(x_l, \omega)h_0(\omega|x_{l-1}, x_l)\, d\omega - \pi_{l-1} \right) \\ &\quad + \beta(x_{l-1}, x_l, x_{l+1}) \\ &\quad \times \left(\int_{\omega: x(\omega) \in [x_l, x_{l+1})} v(x_l, \omega)h_1(\omega|x_l, x_{l+1})\, d\omega - \pi_l \right) = 0, \end{aligned} \tag{2.25}$$

where $\beta(x_{l-1}, x_l, x_{l+1}) \in [0, 1]$ is given by

$$\left(\int_{\omega: x(\omega) \in [x_l, x_{l+1})} \bar{\lambda}(\omega|x_l, x_{l+1})h(\omega|x_l)\, d\omega \right) \tag{2.26}$$

$$\times \left(\int_{\omega:x(\omega)\in[x_{l-1},x_l)} (1-\bar{\lambda}(\omega|x_{l-1},x_l))h(\omega|x_l)\,d\omega \right.$$
$$\left. + \int_{\omega:x(\omega)\in[x_l,x_{l+1})} \bar{\lambda}(\omega|x_l,x_{l+1})h(\omega|x_l)\,d\omega \right)^{-1},$$

and where

$$h_0(\omega|x_{l-1},x_l) = \frac{(1-\bar{\lambda}(\omega|x_{l-1},x_l))h(\omega|x_l)}{\int_{\omega:x(\omega)\in[x_{l-1},x_l)}(1-\bar{\lambda}(\omega|x_{l-1},x_l))h(\omega|x_l)\,d\omega}$$

and

$$h_1(\omega|x_l,x_{l+1}) = \frac{\bar{\lambda}(\omega|x_l,x_{l+1})h(\omega|x_l)}{\int_{\omega:x(\omega)\in[x_l,x_{l+1})}\bar{\lambda}(\omega|x_l,x_{l+1})h(\omega|x_l)\,d\omega}.$$

Note that for $k=0,1$ and $l=1,\dots,L$, $h_k(\cdot|x_{l+k-1},x_{l+k})$ is a continuous density on $\{\omega : x(\omega)\in[x_{l+k-1},x_{l+k})\}$. Note also that $\beta(x_{l-1},x_l,x_{l+1})\in(0,1)$ for $l\in\{2,\dots,L-1\}$ because both integrals in the denominator of (2.26) are positive because the intervals over which they are integrated are nondegenerate.

Thus, whenever the grid size is small enough, (2.25) must hold in every equilibrium. We now employ Lemma 2.6 to show that as Δ, the fineness of the grid, tends to zero, every double-auction equilibrium $b(\cdot)$ for $\mathcal{E}(\alpha,v,f,g,\Delta)$ converges uniformly to the essentially unique symmetric double-auction equilibrium in the market with a fraction α of buyers and in which agents can bid any nonnegative real number.

PROPOSITION 2.7: *For each $\Delta>0$, let $b^\Delta(\cdot)$ be a double-auction equilibrium for $\mathcal{E}(\alpha,v,f,g,\Delta)$. Then $b^\Delta(x)\to v(x,\omega(x))$ uniformly on $[x(0),x(1))$ as $\Delta\to 0$, where $\omega(x)$ is the state ω such that $F(x|\omega)=\alpha$. Also, $P^\Delta(\omega)\to v(x(\omega),\omega)$ uniformly on $[0,1)$, and so the market-clearing price function converges uniformly to the unique fully revealing rational expectations equilibrium of the limit economy $\mathcal{E}(\alpha,v,f,g)$ with a continuum of agents and prices.*

PROOF: Given $b^\Delta(\cdot)$, let $\pi_0^\Delta,\dots,\pi_{L^\Delta}^\Delta$ and $x_0^\Delta<x_1^\Delta<\cdots<x_{L^\Delta+1}^\Delta$ be as in (2.21). Then (2.25) must hold when $(\mathbf{x},\pi)=(\mathbf{x}^\Delta,\pi^\Delta)$ for all $l=1,2,\dots,L^\Delta$.

Let $\bar{x}^\Delta$ be a sequence of signals in $[x(0),x(1))$ such that $\bar{x}^\Delta\to\bar{x}\in[x(0),x(1)]$ and let $\pi_{l^\Delta}^\Delta=b^\Delta(\bar{x}^\Delta)$. Then $[x_{l^\Delta}^\Delta,x_{l^\Delta+1}^\Delta)$ is the interval on which $b^\Delta(\cdot)$ is $\pi_{l^\Delta}^\Delta$ and $\bar{x}^\Delta\in[x_{l^\Delta}^\Delta,x_{l^\Delta+1}^\Delta)$ for every $\Delta>0$. It suffices to show that $b^\Delta(\bar{x}^\Delta)\to v(\bar{x},\omega(\bar{x}))$ as $\Delta\to 0$.

For all Δ, the intersection $[x_{l^\Delta}^\Delta,x_{l^\Delta+1}^\Delta)\cap[x(0),x(1))$ is nonempty because it contains $\bar{x}^\Delta$. Therefore, $x_{l^\Delta+1}^\Delta>x(0)$ and $x_{l^\Delta}^\Delta<x(1)$, but because, by (2.21), $x_1^\Delta\le x(0)$ and $x_{L^\Delta}^\Delta\ge x(1)$, we must then have $0<l^\Delta<L^\Delta$ for all Δ. Hence, the

interval $[x^{\Delta}_{l^{\Delta}}, x^{\Delta}_{l^{\Delta}+1})$ has two adjacent such intervals, one to the left and one to the right, for all Δ. These adjacent intervals are $[x^{\Delta}_{l^{\Delta}-1}, x^{\Delta}_{l^{\Delta}})$ and $[x^{\Delta}_{l^{\Delta}+1}, x^{\Delta}_{l^{\Delta}+2})$. By Lemma 2.6, the difference between any two of $x^{\Delta}_{l^{\Delta}-1}, x^{\Delta}_{l^{\Delta}}$, and $x^{\Delta}_{l^{\Delta}+1}$ converges to zero. Consequently, because $\bar{x}^{\Delta} \in [x^{\Delta}_{l^{\Delta}}, x^{\Delta}_{l^{\Delta}+1})$, each of $x^{\Delta}_{l^{\Delta}-1}, x^{\Delta}_{l^{\Delta}}$, and $x^{\Delta}_{l^{\Delta}+1}$ converges to $\bar{x}$ as $\Delta \to 0$.

Consider now, for each Δ, the two equations in (2.25) for $l = l^{\Delta}$ and $l = l^{\Delta} + 1$ when $(\mathbf{x}, \pi) = (\mathbf{x}^{\Delta}, \pi^{\Delta})$. Consider the limit of these two equations as $\Delta \to 0$, extracting a subsequence along which all of the (finitely many) limiting variables within the two equations converge. In particular, suppose that $\beta(x_{l^{\Delta}-1}, x_{l^{\Delta}}, x_{l^{\Delta}+1}) \to \beta'$, $\beta(x_{l^{\Delta}}, x_{l^{\Delta}+1}, x_{l^{\Delta}+2}) \to \beta''$, $\pi^{\Delta}_{l^{\Delta}-1} \to \pi$, $\pi^{\Delta}_{l^{\Delta}} \to \pi'$, and $\pi^{\Delta}_{l^{\Delta}+1} \to \pi''$. Then, because $x^{\Delta}_{l^{\Delta}-1}, x^{\Delta}_{l^{\Delta}}$, and $x^{\Delta}_{l^{\Delta}+1}$ each converge to $\bar{x}$ as $\Delta \to 0$ and $v(\cdot, \cdot)$ is continuous, the limits of the two equations, $l = l^{\Delta}$ and $l = l^{\Delta} + 1$, are, respectively,

$$(1 - \beta')\big(v(\bar{x}, \omega(\bar{x})) - \pi\big) + \beta'\big(v(\bar{x}, \omega(\bar{x})) - \pi'\big) = 0$$

and

$$(1 - \beta'')\big(v(\bar{x}, \omega(\bar{x})) - \pi'\big) + \beta''\big(v(\bar{x}, \omega(\bar{x})) - \pi''\big) = 0,$$

where $\pi \leq \pi' \leq \pi''$ and $\beta', \beta'' \in [0, 1]$. The first of these equations implies that $v(\bar{x}, \omega(\bar{x})) \leq \pi'$, while the second implies that $v(\bar{x}, \omega(\bar{x})) \geq \pi'$. So, $v(\bar{x}, \omega(\bar{x})) = \pi'$. Because $b^{\Delta}(\bar{x}^{\Delta}) = \pi^{\Delta}_{l^{\Delta}}$, we may conclude that

$$\text{(2.27)} \qquad \lim_{\Delta \to 0} b^{\Delta}(\bar{x}^{\Delta}) = \lim_{\Delta \to 0} \pi^{\Delta}_{l^{\Delta}} = \pi' = v(\bar{x}, \omega(\bar{x})).$$

Because (2.27) holds along every convergent subsequence, it holds for the original sequence as well.

Finally, because $P^{\Delta}(\omega) = b^{\Delta}(x(\omega))$ for every $\omega \in [0, 1)$, (2.27) implies that $P^{\Delta}(\omega) \to v(x(\omega), \omega)$ uniformly in ω on $[0, 1)$ as $\Delta \to 0$. *Q.E.D.*

The next lemma states that for each $\Delta > 0$ small enough and generic, the lengths of the intervals over which an equilibrium bidding function assumes the highest and lowest equilibrium market-clearing prices are bounded away from zero uniformly across equilibria of $\mathcal{E}(\alpha, v, f, g, \Delta)$ (see Lemma 2.8 (c) and (d)). Moreover, if $[\underline{x}, \bar{x})$ is the interval over which an equilibrium bidding function is the lowest market-clearing equilibrium price $\underline{p}$, then $v(\underline{x}, 0) - \underline{p} + \Delta$ is strictly positive. Indeed, this difference is bounded away from zero across all equilibria of $\mathcal{E}(\alpha, v, f, g, \Delta)$; a similar result holds for the highest market-clearing price (see Lemma 2.8 (a) and (b)). These properties will be important later on when we construct an equilibrium for the large finite economy. This is the only place where we directly employ RP Assumption A.4, namely that $v_{\omega}(x, 0) = v_{\omega}(x, 1) = 0$. (Of course, RP Assumption A.4 is indirectly employed whenever we appeal to the lemma we are about to prove.)

LEMMA 2.8: *For all $\Delta > 0$ sufficiently small and such that neither $v(x(0), 0)$ nor $v(x(1), 1)$ is an integer multiple of Δ, there exists $\bar{\varepsilon} > 0$ such that, for all $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$, the following statements hold*:

(a) $v(\hat{x}_{k_1}, 0) > (k_1 - 1)\Delta + \bar{\varepsilon}$;
(b) $v(\hat{x}_{k_L}, 1) < k_L\Delta - \bar{\varepsilon}$;
(c) $\hat{x}_{k_2} - \hat{x}_{k_1} > \bar{\varepsilon}$;
(d) $\hat{x}_{k_L} - \hat{x}_{k_{L-1}} > \bar{\varepsilon}$.

Here $k_1\Delta < \cdots < k_{L-1}\Delta$ is the range of $P(\cdot) \equiv b_{\hat{\mathbf{x}}}(x(\cdot))$ and $k_L \equiv k_{L-1} + 1$.

PROOF: We prove (a) and (c) only because (b) and (d) are similar. We begin with (a).

Because the set of fixed points of $\operatorname{co} B(\cdot)$ is compact for each Δ and because $v(x, 0)$ is continuous, it suffices to show that for all $\Delta > 0$ sufficiently small and for all $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$,

$$v(\hat{x}_{k_1}, 0) > (k_1 - 1)\Delta, \tag{2.28}$$

where the range of $P(\cdot) \equiv b_{\hat{\mathbf{x}}}(x(\cdot))$ is $k_1\Delta < \cdots < k_{L-1}\Delta$.

Consider then an arbitrary sequence $\Delta \to 0^+$ and for each Δ along the sequence consider an arbitrary fixed point $\hat{\mathbf{x}}^\Delta \in \operatorname{co} B(\hat{\mathbf{x}}^\Delta)$. Let $k_1^\Delta\Delta < \cdots < k_{L^\Delta-1}^\Delta\Delta$ be the range of $P^\Delta(\cdot) \equiv b_{\hat{\mathbf{x}}^\Delta}(x(\cdot))$. For $l = 1, \ldots, L-1$ and all Δ, let $x_l^\Delta = \hat{x}_{k_l^\Delta}^\Delta$ and $\pi_l^\Delta = k_l^\Delta\Delta$. Without loss, we may assume that $\lim_\Delta x_l^\Delta$ and $\lim_\Delta \pi_l^\Delta$ exist for all l (because $x_l^\Delta \in [0, 1]$ and $0 \leq \pi_l^\Delta \leq v(1, 1) + \Delta$). It suffices to show that (2.28) holds when $\hat{\mathbf{x}} = \mathbf{x}^\Delta$ for all Δ sufficiently small.

For Δ sufficiently small, by Proposition 2.3, $b_{\hat{\mathbf{x}}^\Delta}(\cdot)$ is a double-auction equilibrium of $\mathcal{E}(\alpha, v, f, g, \Delta)$, and so by (2.21) and Lemma 2.6, again for sufficiently small Δ, we have $L \geq 4$ and $0 < x_1^\Delta \leq x(0) < x_2^\Delta < x_3^\Delta < x(1)$. For convenience, we assume without loss that these conclusions hold for all Δ. Note also that Lemma 2.6 implies that both x_1^Δ and x_2^Δ converge to $x(0)$.

Now, because $b_{\hat{\mathbf{x}}^\Delta}(\cdot)$ is an equilibrium, (2.25) must hold. For $l = 1$ and 2, we therefore obtain, for every Δ,

$$\int_0^{\omega(x_2^\Delta)} v(x_1^\Delta, \omega) h_1(\omega | x_1^\Delta, x_2^\Delta)\, d\omega = \pi_1^\Delta \tag{2.29}$$

and

$$\begin{aligned}(1 - \beta(x_1^\Delta, x_2^\Delta, x_3^\Delta))&\left(\int_0^{\omega(x_2^\Delta)} v(x_2^\Delta, \omega) h_0(\omega | x_1^\Delta, x_2^\Delta)\, d\omega - \pi_1^\Delta\right) \\ &+ \beta(x_1^\Delta, x_2^\Delta, x_3^\Delta)\left(\int_{\omega(x_2^\Delta)}^{\omega(x_3^\Delta)} v(x_2^\Delta, \omega) h_1(\omega | x_2^\Delta, x_3^\Delta)\, d\omega - \pi_2^\Delta\right) = 0,\end{aligned} \tag{2.30}$$

where $\beta(x_1^\Delta, x_2^\Delta, x_3^\Delta) \in (0,1)$ (see the discussion following (2.25)) and each of the $h_k(\cdot|x_l^\Delta, x_{l+1}^\Delta)$ are densities on the intervals over which the integrals in which they appear are integrated. We next wish to establish the two inequalities

$$v(x_1^\Delta, \omega(x_2^\Delta)) \geq \pi_1^\Delta \tag{2.31}$$

and

$$v(x_2^\Delta, \omega(x_2^\Delta)) - \pi_1^\Delta < \Delta. \tag{2.32}$$

Inequality (2.31) follows immediately from (2.29) because $v(x, \omega)$ is nondecreasing in ω. For (2.32), note that (2.29) implies that the first integral in parentheses in (2.30) is strictly positive. This is because $v(x, \omega)$ is strictly increasing in x and nondecreasing in ω, $x_2^\Delta > x_1^\Delta$, and $h_0(\cdot|x_1^\Delta, x_2^\Delta)$ first-order stochastically dominates $h_1(\cdot|x_1^\Delta, x_2^\Delta)$.[12] Consequently, because $\beta(x_1^\Delta, x_2^\Delta, x_3^\Delta) \in (0,1)$, the second integral in parentheses in (2.30) is strictly negative. This has two implications. First, because $v(x, \omega)$ is nondecreasing in ω, we must have $v(x_2^\Delta, \omega(x_2^\Delta)) - \pi_2^\Delta < 0$. Second, and less obvious, is that there can be no price in the grid strictly between π_1^Δ and π_2^Δ. If there were such a price π', a buyer with signal x_2^Δ would be strictly better off bidding π' rather than π_2^Δ because both bids win (resp., do not win) a unit when the market-clearing price is π_1^Δ (resp., above π_2^Δ), but only the bid π_2^Δ wins a unit with positive probability and earns a negative expected payoff (because the second integral in parentheses in (2.30) is strictly negative) when the market-clearing price is π_2^Δ. However, this yields a contradiction because, according to the equilibrium, $b_{\hat{\mathbf{x}}^\Delta}(x_2^\Delta) = \pi_2^\Delta$. Hence, there is no price in the grid strictly between π_1^Δ and π_2^Δ. Because $\pi_2^\Delta > \pi_1^\Delta$, we conclude that $\pi_2^\Delta - \pi_1^\Delta = \Delta$. Combining this with $v(x_2^\Delta, \omega(x_2^\Delta)) - \pi_2^\Delta < 0$, we obtain (2.32).

Finally, we wish to establish that for all Δ sufficiently small,

$$\frac{v(x_2^\Delta, \omega(x_2^\Delta)) + v(x_1^\Delta, 0)}{2} > v(x_1^\Delta, \omega(x_2^\Delta)). \tag{2.33}$$

Before we establish (2.33), note that if it holds, then

$$\begin{aligned}\frac{\pi_1^\Delta + \Delta + v(x_1^\Delta, 0)}{2} &> \frac{v(x_2^\Delta, \omega(x_2^\Delta)) + v(x_1^\Delta, 0)}{2} \\ &> v(x_1^\Delta, \omega(x_2^\Delta)) \\ &\geq \pi_1^\Delta,\end{aligned}$$

[12]First-order stochastic dominance follows because $h_0(\omega|x_1^\Delta, x_2^\Delta)/h_1(\omega|x_1^\Delta, x_2^\Delta)$ is nondecreasing in ω, which itself follows because $\bar{\lambda}(\omega|x_1^\Delta, x_2^\Delta)$ is nonincreasing in ω, and $f(x|\omega)$ (and hence $h(\omega|x)$) satisfies the MLRP.

where the first and third lines follow from (2.32) and (2.31), respectively, and the second line follows from (2.33). Rearranging the outer expressions yields $v(x_1^\Delta, 0) > \pi_1^\Delta - \Delta = (k_1^\Delta - 1)\Delta$, proving (2.28) when $\hat{\mathbf{x}} = \mathbf{x}^\Delta$, as desired. Hence, it remains only to prove (2.33).

Because $x_2^\Delta > x_1^\Delta$, we may rewrite (2.33) as

$$\frac{v(x_2^\Delta, \omega(x_2^\Delta)) - v(x_1^\Delta, \omega(x_2^\Delta))}{x_2^\Delta - x_1^\Delta} + \frac{v(x_1^\Delta, 0) - v(x_1^\Delta, \omega(x_2^\Delta))}{x_2^\Delta - x_1^\Delta} > 0. \tag{2.34}$$

Now, because $v_x(x, \omega)$ is continuous, the first term on the left-hand side of (2.34) converges to $v_x(x(0), 0) > 0$. Consequently, it suffices to show that the second term converges to zero. Because $x_2^\Delta > x(0)$, we have $\omega(x_2^\Delta) > 0$ and so the second term can be written as the product

$$\frac{\omega(x_2^\Delta)}{x_2^\Delta - x(0)} \frac{x_2^\Delta - x(0)}{x_2^\Delta - x_1^\Delta} \frac{v(x_1^\Delta, 0) - v(x_1^\Delta, \omega(x_2^\Delta))}{\omega(x_2^\Delta)}.$$

Because $x(0) > 0$, the strict MLRP implies that $F_\omega(x(0)|0) < 0$. Therefore, the limit of the first term in the preceding product exists and is $-F_x(x(0)|0)/F_\omega(x(0)|0)$.[13] The second term is bounded; indeed, it is in $[0, 1]$ because $x_2^\Delta > x(0) > x_1^\Delta$. Finally, because $v_\omega(x, \omega)$ is continuous and $\omega(x_2^\Delta) \to \omega(x(0)) = 0$, the third term converges to $-v_\omega(x(0), 0)$, which is zero by RP Assumption A.4. This proves (2.33).

We now prove (c). By what we have just shown, we may choose $\bar{\Delta} > 0$ according to Proposition 2.3 and also so that (a) holds. Fix any $\Delta < \bar{\Delta}$ such that $k\Delta \neq v(x(0), 0)$ for all $k = 0, 1, \ldots$. Let $\{\hat{\mathbf{x}}\}$ be a sequence of fixed points of $\operatorname{co} B(\cdot)$, and suppose without loss (because $\bar{\mathcal{P}}$ is finite given Δ) that the range of $P(\cdot) \equiv b_{\hat{\mathbf{x}}}(x(\cdot))$ is constant and equal to $\{k_1\Delta, \ldots, k_{L-1}\Delta\}$ along the sequence. If, contrary to (c), $|\hat{x}_{k_2} - \hat{x}_{k_1}| \to 0$, then because, by (2.21), $\hat{x}_1 \leq x(0) < \hat{x}_{k_2}$, we must have $\hat{x}_{k_1} \to x(0)$ and $\hat{x}_{k_2} \to x(0)$. However, taking the limit of (2.29), which must hold because $b_{\hat{\mathbf{x}}}(\cdot)$ is an equilibrium by Proposition 2.3, we obtain $v(x(0), 0) = k_1\Delta$, a contradiction, thus proving (c).

We have shown therefore that there is a single $\bar{\Delta}$ such that for every $\Delta < \bar{\Delta}$, (a) holds for some $\bar{\varepsilon}$ and (c) holds for some possibly distinct $\bar{\varepsilon}$. Clearly, (a) and (c) hold simultaneously for the smaller of the two epsilons. *Q.E.D.*

In the course of proving Lemma 2.8, we showed that for Δ sufficiently small, $k_2\Delta \geq v(\hat{x}_{k_2}, \omega(\hat{x}_{k_2}))$ and $k_2 = k_1 + 1$ for all $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$. Consequently, because $\hat{x}_{k_2} > x(0)$, $v_x > 0$, and $v_\omega \geq 0$, we obtain $(k_1 + 1)\Delta > v(x(0), 0)$. Similarly, it can be shown that $(k_{L-1} - 1)\Delta < v(x(1), 1)$. We record these results for future reference.

[13] This can be derived from the fact that, by the definition of $\omega(\cdot)$, $F(x_2^\Delta|\omega(x_2^\Delta)) = \alpha$ is constant for all Δ .

LEMMA 2.9: *For all* $\Delta > 0$ *sufficiently small and for all* $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$, (a) $(k_1 + 1)\Delta > v(x(0), 0)$ *and* (b) $(k_{L-1} - 1)\Delta < v(x(1), 1)$, *where* $k_1\Delta < \cdots < k_{L-1}\Delta$ *is the range of* $P(\cdot) \equiv b_{\hat{\mathbf{x}}}(x(\cdot))$.

2.2. *Part B*

As we have seen, when Δ is small enough, every double-auction equilibrium for $\mathcal{E}(\alpha, v, f, g, \Delta)$ yields a vector of signals $0 < x_1 \leq x(0) < x_2 < \cdots < x_{L-1} < x(1) \leq x_L < 1$ and corresponding prices $0 < \pi_1 < \cdots < \pi_{L-1} < K\Delta$ in $\bar{\mathcal{P}}$ that satisfy (2.25). We now investigate certain genericity properties of such systems of equations. To do so, we consider perturbations of both the value function v and the fineness of the grid Δ.

Suppose we replace $v(x, \omega)$ in (2.25) with the function $v(x, \omega) + \varepsilon_1 + \varepsilon_2 x + \varepsilon_3 x^2 + \cdots + \varepsilon_L x^{L-1}$ for some $\varepsilon_1, \ldots, \varepsilon_L \in \mathbb{R}$. (It is not important that the resulting function need not lie in V.) The system (2.25) then becomes, for $l = 1, \ldots, L$,

$$
\begin{aligned}
(2.35)\qquad & (1 - \beta(x_{l-1}, x_l, x_{l+1})) \\
& \times \left(\int_{\omega : x(\omega) \in [x_{l-1}, x_l)} v(x_l, \omega) h_0(\omega | x_{l-1}, x_l)\, d\omega - \pi_{l-1} \right) \\
& + \beta(x_{l-1}, x_l, x_{l+1}) \left(\int_{\omega : x(\omega) \in [x_l, x_{l+1})} v(x_l, \omega) h_1(\omega | x_l, x_{l+1})\, d\omega - \pi_l \right) \\
& + \varepsilon \cdot \mathbf{p}_L(x_l) = 0,
\end{aligned}
$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_L)$ and $\mathbf{p}_L(x) = (1, x, x^2, \ldots, x^{L-1})$, and where we define $x_0 = \pi_0 = 0$, $x_{L+1} = 1$, and $\pi_L = K\Delta$.

Recalling that each $\pi_l = k_l \Delta$ for some nonnegative integer k_l, the following perspective will be useful. For any $L \geq 2$ and any sequence of nonnegative integers $k_0 < k_1 < k_2 < \cdots < k_L$, replacing each π_l by $k_l\Delta$, we can view the L equations in (2.35) as a system of equations in the variables $x_1, \ldots, x_L$, $\varepsilon_1, \ldots, \varepsilon_L$, and Δ, while holding fixed $L, k_0, k_1, \ldots,$ and k_L. Let

$$(2.36)\qquad \Phi(\mathbf{x}, \varepsilon, \Delta) = 0$$

denote this system of equations, where $\mathbf{x} = (x_1, \ldots, x_L)$, and the zero on the right-hand side is an L vector. We maintain the assumption that $L \geq 2$ for the remainder of this section.

Let U denote the open set of strictly increasing vectors, $(x_1, \ldots, x_L) \in (0, 1)^L$ such that $x_2 > x(0)$ and $x_{L-1} < x(1)$. Then $\Phi(\mathbf{x}, \varepsilon, \Delta)$ is well defined on $U \times \mathbb{R}^{L+1}$. Note that in addition to negative vectors ε, we allow here $\Delta \leq 0$, and also $x_1 > x(0)$ and $x_L < x(1)$. This is convenient for the formal analysis of the system of equations Φ.

The function $\Phi(\mathbf{x}, \varepsilon, \Delta)$ is linear in ε and its derivative with respect to ε is the matrix

$$\Phi_\varepsilon(\mathbf{x}, \varepsilon, \Delta) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{L-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{L-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_L & x_L^2 & \cdots & x_L^{L-1} \end{pmatrix}, \tag{2.37}$$

which has full rank because the x_l are distinct.[14]

Hence, for every $\mathbf{x} \in U$ and every $\Delta \in \mathbb{R}$, there is a unique $\varepsilon \in \mathbb{R}^L$ that solves (2.35). Let us denote this solution by $\varepsilon(\mathbf{x}, \Delta)$.

Now, although $\Phi(\mathbf{x}, \varepsilon, \Delta)$ is differentiable in ε, it is not differentiable in $\mathbf{x}$ on U. (Hence, $\varepsilon(\mathbf{x}, \Delta)$ is not differentiable either.) In fact, differentiability fails when $x_1 = x(0)$ or $x_L = x(1)$.[15] To see this for $x_1 = x(0)$, note that when $l = 1$, the left-hand side of (2.35) involves integration over the set of states ω such that $x(\omega) \in [0, x_1)$ and $x(\omega) \in [x_1, x_2)$. Now, if $x_1 < x(0)$, then the first set of states is empty and the second is locally independent of x_1. Consequently, the derivatives of the corresponding integrals do not depend on the limits of integration, but if $x_1 > x(0)$, both sets are nonempty and both vary with x_1. Moreover, this variation affects the derivative in a manner that is bounded away from zero. Hence, differentiability at $x_1 = x(0)$ fails. Similarly, differentiability at $x_L = x(1)$ fails.

These are the only points in U at which $\Phi(\mathbf{x}, \varepsilon, \Delta)$ fails to be differentiable in $\mathbf{x}$. Indeed, it is otherwise continuously differentiable in all variables. Fortunately, for every $\Delta > 0$, the set of value perturbations ε that admit solutions $\mathbf{x}$ to $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ such that $x_1 = x(0)$ or $x_L = x(1)$ are rare. In fact, this set has Lebesgue measure zero, as we now establish.

LEMMA 2.10: *Fix any $\Delta \in \mathbb{R}$. For almost every $\varepsilon \in \mathbb{R}^L$, if $\mathbf{x} \in U$ solves $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$, then $x_1 \neq x(0)$ and $x_L \neq x(1)$.*

PROOF: Fix $\Delta \in \mathbb{R}$ and denote the function $\varepsilon(\mathbf{x}, \Delta)$ simply by $\varepsilon(\mathbf{x})$. We must show that the set of $\varepsilon \in \mathbb{R}^L$ for which $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ possesses a solution $\mathbf{x} \in U$ such that either (a) $x_1 = x(0)$ and $x_L = x(1)$, (b) $x_1 = x(0)$ and $x_L \neq x(1)$, or (c) $x_1 \neq x(0)$ and $x_L = x(1)$ has Lebesgue measure zero.

We provide the argument for case (a) only; the other two cases are similar. Fix $x_1 = x(0)$ and $x_L = x(1)$, and consider $\varepsilon(x(0), x_2, \ldots, x_{L-1}, x(1))$ as a function that maps vectors $\mathbf{z} = (x_2, \ldots, x_{L-1})$ such that $x(0) < x_2 < \cdots < x_{L-1} < x(1)$ into $\mathbb{R}^L$. Note that if $L = 2$, then $\mathbf{z}$ is the null vector and $\varepsilon(x(0), x_2, \ldots, x_{L-1}, x(1)) = \varepsilon(x(0), x(1))$ is constant.

[14]The matrix maps a nonzero vector $(a_0, a_1, \ldots, a_{L-1})$ to the zero vector if and only if each x_l is a root of the nonconstant polynomial $a_0 + a_1 x + \cdots + a_{L-1} x^{L-1}$. However, this polynomial has at most $L - 1$ distinct roots and so this would imply that the x_l are not distinct.

[15]Recall that because $f > 0$, $0 < x(0) < x(1) < 1$.

Note that the range of $\varepsilon(x(0), \mathbf{z}, x(1))$ gives precisely the set of perturbations ε for which $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ possesses a solution $\mathbf{x}$, with $x_1 = x(0)$ and $x_L = x(1)$. To take care of case (a), it suffices to show that this range has measure zero in $\mathbb{R}^L$.

Because $\Phi(x(0), \mathbf{z}, x(1), \varepsilon, \Delta)$ is continuously differentiable in the arguments $\mathbf{z}, \varepsilon, \Delta$, (recall that $x(0) < x_2 < \cdots < x_{L-1} < x(1)$) and because, as already established, Φ_ε has full rank, the implicit function theorem implies that $\varepsilon(x(0), \mathbf{z}, x(1))$ is continuously differentiable in $\mathbf{z}$ on its domain.

Now, the derivative $\varepsilon_{\mathbf{z}}(x(0), \mathbf{z}, x(1))$ has rank at most $L - 2$, the number of coordinates of $\mathbf{z}$. Hence, because $\varepsilon(x(0), \mathbf{z}, x(1))$ takes values in $\mathbb{R}^L$, every value of $\varepsilon(x(0), \mathbf{z}, x(1))$ is critical, by definition. By Sard's theorem, the set of critical values of $\varepsilon(x(0), \mathbf{z}, x(1))$ has measure zero in $\mathbb{R}^L$. We conclude that the range of $\varepsilon(x(0), \mathbf{z}, x(1))$, as $\mathbf{z}$ varies over its domain, has measure zero in $\mathbb{R}^L$ as desired. *Q.E.D.*

Lemma 2.10 has the following immediate implication.

LEMMA 2.11: *Let $U^0 = \{\mathbf{x} \in U : x_1 \neq x(0)$ and $x_L \neq x(1)\}$. Then $\Phi(\mathbf{x}, \varepsilon, \Delta)$ is continuously differentiable on $U^0 \times \mathbb{R}^L \times \mathbb{R}$ and, for every $\Delta \in \mathbb{R}$, there is a subset C of $\mathbb{R}^L$ whose complement has Lebesgue measure zero such that for every $\varepsilon \in C$, all solutions $\mathbf{x} \in U$ to $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ are in U^0.*

PROOF: Continuous differentiability of $\Phi(\mathbf{x}, \varepsilon, \Delta)$ on $U^0 \times \mathbb{R}^{L+1}$ follows from the continuous differentiability of the functions involved in its definition, and the measure zero result follows directly from Lemma 2.10. *Q.E.D.*

Having established that it is rarely the case that solutions $\mathbf{x}$ to $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ involve either $x_1 = x(0)$ or $x_L = x(1)$, we now consider another possible property of these solutions and establish that it too rarely obtains. Specifically, we wish to establish that if an agent with signal x_l is indifferent, in equilibrium, between bidding π_{l-1} and π_l, then, typically, the agent's expected payoff from bidding the higher price π_l is not zero, conditional on (i) his signal, (ii) π_l being the market-clearing price, and (iii) ending up with the good subsequent to whatever rationing takes place.

The following lemma provides a precise statement. Recall that $\pi_l = k_l \Delta$ for some nonnegative integers $k_0 < \cdots < k_L$ and that, by definition,

$$h_1(\omega|x_l, x_{l+1}) = \frac{\bar{\lambda}(\omega|x_l, x_{l+1}) h(\omega|x_l)}{\int_{\omega: x(\omega) \in [x_l, x_{l+1})} \bar{\lambda}(\omega|x_l, x_{l+1}) h(\omega|x_l)\, d\omega}$$

is the agent's posterior density of the state ω, conditional on (i) the agent's signal x_l, (ii) the market-clearing price being π_l (i.e., $x(\omega) \in [x_l, x_{l+1})$), and (iii) the agent ending up with good subsequent to any rationing.

LEMMA 2.12: *For almost every* $(\varepsilon, \Delta) \in \mathbb{R}^{L+1}$, *every solution* $\mathbf{x} \in U$ *to* $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ *is such that* $x_1 \neq x(0)$, $x_L \neq x(1)$, *and*

$$\int_{\omega: x(\omega)\in[x_l, x_{l+1})} v(x_l, \omega) h_1(\omega | x_l, x_{l+1})\, d\omega + \varepsilon \cdot \mathbf{p}_L(x_l) - k_l \Delta \neq 0 \tag{2.38}$$

for every l *such that* $x_l \in (x(0), x(1))$.

PROOF: By Lemma 2.11 and Fubini's theorem, there is a Borel subset E of $\mathbb{R}^{L+1}$, whose complement has Lebesgue measure zero, such that for every $(\varepsilon^0, \Delta^0) \in E$, every solution $\mathbf{x}^0 \in U$ to $\Phi(\mathbf{x}, \varepsilon^0, \Delta^0) = 0$ is such that $\mathbf{x}^0 \in U^0$. It therefore suffices to establish (2.38) for all $\mathbf{x} \in U^0$. Note that $\Phi(\mathbf{x}, \varepsilon, \Delta)$ is continuously differentiable on $U^0 \times \mathbb{R}^{L+1}$.

Given $l = 1, \dots, L$, consider modifying the system of L equations $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ by replacing its lth equation by the *two* equations

$$\int_{\omega: x(\omega)\in[x_{l-1}, x_l)} v(x_l, \omega) h_0(\omega | x_{l-1}, x_l)\, d\omega + \varepsilon \cdot \mathbf{p}_L(x_l) - k_{l-1} \Delta = 0 \tag{2.39}$$

and

$$\int_{\omega: x(\omega)\in[x_l, x_{l+1})} v(x_l, \omega) h_1(\omega | x_l, x_{l+1})\, d\omega + \varepsilon \cdot \mathbf{p}_L(x_l) - k_l \Delta = 0. \tag{2.40}$$

Denote this new system of $L + 1$ equations by $\Psi^l(\mathbf{x}, \varepsilon, \Delta)$. Note that the left-hand sides of (2.40) and (2.38) are identical.

The lth equation removed from $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ is a strict convex combination of the two equations (2.39) and (2.40) whenever $x_l \in (x(0), x(1))$. Consequently, $x_l \in (x(0), x(1))$ and $\Psi^l(\mathbf{x}, \varepsilon, \Delta) = 0$ imply $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$. In addition, the strict convex combination means that (2.38) can fail for some $\mathbf{x}$ such that $\Phi(\mathbf{x}, \varepsilon, \Delta) = 0$ and such that $x_l \in (x(0), x(1))$ only if both (2.40) and (2.39) hold, and so only if $\Psi^l(\mathbf{x}, \varepsilon, \Delta) = 0$. Hence, it suffices to show that, for each $l = 1, 2, \dots, L$ and for almost every $(\varepsilon, \Delta) \in \mathbb{R}^{L+1}$, the system $\Psi^l(\mathbf{x}, \varepsilon, \Delta) = 0$ has no solution $\mathbf{x} \in U^0$.

For any fixed $\mathbf{x}^0 \in U^0$, we first argue that there is a unique $(\varepsilon^0, \Delta^0)$ such that $\Psi^l(\mathbf{x}^0, \varepsilon^0, \Delta^0) = 0$. To see this, note that, given $\mathbf{x}^0$, any such $(\varepsilon^0, \Delta^0)$ must satisfy (2.40) and (2.39). Subtracting either one of these equations from the other eliminates ε and yields an equation in Δ alone. Because $k_{l-1} < k_l$, there is a unique Δ that solves this equation and so we may write $\Delta^0 = \Delta(\mathbf{x}^0)$. Moreover, again because $k_{l-1} < k_l$, the function $\Delta(\mathbf{x})$ is continuously differentiable. Now, because $\mathbf{x}^0$ solves $\Psi^l(\mathbf{x}, \varepsilon^0, \Delta^0) = 0$, it must also solve $\Phi(\mathbf{x}, \varepsilon^0, \Delta^0) = 0$. However, this means that $\varepsilon^0 = \varepsilon(\mathbf{x}^0, \Delta^0) = \varepsilon(\mathbf{x}^0, \Delta(\mathbf{x}^0))$. Letting $\bar{\varepsilon}(\mathbf{x}) = \varepsilon(\mathbf{x}, \Delta(\mathbf{x}))$, we have that $\bar{\varepsilon}(\mathbf{x})$ is continuously differentiable on U^0, being the composition of such functions.

Therefore, for each $\mathbf{x} \in U^0$, $(\bar{\varepsilon}(\mathbf{x}), \Delta(\mathbf{x}))$ is the unique solution (ε, Δ) to $\Psi^l(\mathbf{x}, \varepsilon, \Delta) = 0$. Consequently, the range of $(\bar{\varepsilon}(\mathbf{x}), \Delta(\mathbf{x}))$ as $\mathbf{x}$ varies throughout U^0 is the entire set of (ε, Δ) such that $\Psi^l(\mathbf{x}, \varepsilon, \Delta) = 0$ possesses a solution $\mathbf{x} \in U^0$. It suffices then to demonstrate that the range of $(\bar{\varepsilon}(\mathbf{x}), \Delta(\mathbf{x}))$ has Lebesgue measure zero in $\mathbb{R}^{L+1}$.

The function $(\bar{\varepsilon}(\mathbf{x}), \Delta(\mathbf{x}))$ is a continuously differentiable map from the open set $U^0 \subset \mathbb{R}^L$ into $\mathbb{R}^{L+1}$. Consequently, the rank of its derivative $(\bar{\varepsilon}_{\mathbf{x}}(\mathbf{x}), \Delta_{\mathbf{x}}(\mathbf{x}))$ is at most L for every $\mathbf{x} \in U^0$. Therefore, by definition, each value in the range of $(\bar{\varepsilon}(\mathbf{x}), \Delta(\mathbf{x}))$ is critical. By Sard's theorem, the set of critical values of $(\bar{\varepsilon}(\mathbf{x}), \Delta(\mathbf{x}))$, and hence its entire range, has Lebesgue measure zero, as desired. *Q.E.D.*

Recall from RP that V denotes the subset of functions, $v(x, \omega)$ that satisfy RP Assumptions A.3 and A.4, and that RP define a norm on V by $\|v\| = \max_{x,\omega} v(x, \omega)$, thereby inducing a topology on V.[16]

LEMMA 2.13: *There is a residual subset V^0 of V such that for every $v \in V^0$, $v(0,0) > 0$ and the following scenario holds for a residual set of $\Delta \in \mathbb{R}$: For every positive integer $L \geq 2$ and every strictly increasing sequence of nonnegative integers $k_0 < k_1 < \cdots < k_L$, if $0 = x_0 < x_1 < \cdots < x_L < x_{L+1} = 1$ satisfies $x_2 > x(0)$, $x_{L-1} < x(1)$, and*

$$(2.41)\quad \int_{\omega : x(\omega) \in [x_{l-1}, x_l)} (v(x_l, \omega) - k_{l-1}\Delta)(1 - \bar{\lambda}(\omega | x_{l-1}, x_l)) h(\omega | x_l)\, d\omega + \int_{\omega : x(\omega) \in [x_l, x_{l+1})} (v(x_l, \omega) - k_l \Delta)\bar{\lambda}(\omega | x_l, x_{l+1}) h(\omega | x_l)\, d\omega = 0$$

for each $l = 1, 2, \ldots, L$, then

$$(2.42)\quad x_1 \neq x(0), \quad x_L \neq x(1),$$

and

$$(2.43)\quad \int_{\omega : x(\omega) \in [x_l, x_{l+1})} (v(x_l, \omega) - k_l \Delta)\bar{\lambda}(\omega | x_l, x_{l+1}) h(\omega | x_l)\, d\omega \neq 0$$

for every l such that $x_l \in (x(0), x(1))$.

PROOF: Because $\{v \in V : v(0,0) > 0\}$ is an open and dense subset of V, it suffices to prove the statement while ignoring the condition $v(0,0) > 0$. Let

[16]Any topology on V that is at least this strong and for which linear combinations of elements of V are continuous in the coefficients will do.

$\Phi(\mathbf{x}, v, \Delta, L)$ denote the system (2.36), but where $\varepsilon = 0$ and where we explicitly keep track of $v \in V$ and the dimension $L \geq 2$. Recall that U denotes the open set of strictly increasing vectors $(x_1, \ldots, x_L) \in (0, 1)^L$ such that $x_2 > x(0)$ and $x_{L-1} < x(1)$.

Define $\bar{U}_{L,n}$ to be the compact subset of strictly increasing vectors $\mathbf{x}$ in U such that the distance between any two coordinates of $\mathbf{x}$ is at least $1/n$, and such that $x_1 \geq 1/n$, $x_2 \geq x(0) + 1/n$, $x_{L-1} \leq x(1) - 1/n$, and $x_L \leq 1 - 1/n$. Note that, given L, the union over n of the $\bar{U}_{L,n}$ is U.

Define $A_{L,n} = \{(v, \Delta) \in V \times \mathbb{R}$: every solution $\mathbf{x} \in \bar{U}_{L,n}$ to $\Phi(\mathbf{x}, v, \Delta, L) = 0$ is such that (2.42) and (2.43) hold$\}$. Because $\bar{U}_{L,n}$ is compact, $A_{L,n}$ is open. Moreover, by Lemma 2.12, if $(v, \Delta) \in V \times \mathbb{R}$ is not in $A_{L,n}$, then for some $\varepsilon \in \mathbb{R}_{++}$ arbitrarily close to the origin, and some $\Delta' \in \mathbb{R}$ arbitrarily close to Δ, $(v + \mathbf{p}_L \cdot \varepsilon, \Delta') \in A_{L,n}$. Consequently, $A_{L,n}$ is dense in $V \times \mathbb{R}$. The set $A = \bigcap_{n,L} A_{L,n}$ is therefore residual and has the property that every $(v, \Delta) \in A$ satisfies the conclusion of the lemma. By the Kuratowski–Ulam theorem (see Oxtoby (1980, p. 56, Theorem 15.1)), there exists a residual subset V^0 of V such that for every $v \in V^0$, $\{\Delta \in \mathbb{R} : (v, \Delta) \in A\}$ is a residual subset of $\mathbb{R}$. *Q.E.D.*

2.3. *Part C*

Consider again the economy $\mathcal{E}(\alpha, v, f, g, \Delta)$ with a unit mass of agents, of whom $\alpha \in (0, 1)$ are buyers, and where all agents are restricted to the discrete grid of prices $\mathcal{P} = \{0, \Delta, 2\Delta, \ldots\}$.

Our objective here is to allow buyers and sellers to behave asymmetrically, within a certain range, even though they are symmetric in this continuum agent setting. We shall define a particularly useful correspondence from pairs of asymmetric buyer–seller bid functions, $(b(\cdot), s(\cdot))$ into subsets of them and demonstrate that every fixed point $(\bar{b}(\cdot), \bar{s}(\cdot))$ of this correspondence is such that both $\bar{b}(\cdot)$ and $\bar{s}(\cdot)$ are each (symmetric) double-auction equilibria for $\mathcal{E}(\alpha, v, f, g, \Delta)$. Furthermore, the two equilibria are outcome equivalent.

Because much of what follows is similar to the presentation in Part A, we need not dwell on the details. Let $\bar{\mathcal{P}} = \{0, \Delta, 2\Delta, \ldots, K\Delta\} = \{p_0, p_1, \ldots, p_K\}$, where $K\Delta \geq v(1, 1) > (K - 1)\Delta$. Recall that X_K is the set of nondecreasing vectors in $[0, 1]^K$ and that each of these represents a right-continuous monotone bidding function. Buyers' bidding functions are represented by $\mathbf{x} \in X_K$ and sellers' bidding functions are represented by $\mathbf{y} \in X_K$. For $\varepsilon \geq 0$, let $C_\varepsilon = \{(\mathbf{x}, \mathbf{y}) \in X_K \times X_K : \max_k |x_k - y_k| \leq \varepsilon\}$. Hence, C_ε is a nonempty, compact, convex set.

Consider a buyer (sellers have equivalent preferences). If all other buyers employ $\mathbf{x} \in X_K$ and all sellers employ $\mathbf{y} \in X_K$, respectively, then the buyer's payoff when his signal is x and he bids p is

$$(2.44)\qquad u(p, x|\mathbf{x}, \mathbf{y}) = \int_{\omega:P(\omega)<p} (v(x, \omega) - P(\omega))h(\omega|x)\, d\omega + \int_{\omega:P(\omega)=p} (v(x, \omega) - p)\lambda(\omega|p, \mathbf{x}, \mathbf{y})h(\omega|x)\, d\omega,$$

where $P(\omega)$ is the market-clearing price in state $\omega \in [0, 1]$, and $\lambda(\omega|p, \mathbf{x}, \mathbf{y})$ is the probability that the buyer ends up with the good when $P(\omega) = p$. These latter two functions are determined by the others' strategies $\mathbf{x}$ and $\mathbf{y}$ as follows.

For $k = 0, 1, \ldots, K$, according to the strategy $\mathbf{x}$, a buyer bids p_k when his signal is in $[x_k, x_{k+1})$, and according to the strategy $\mathbf{y}$, a seller bids p_k when his signal is in $[y_k, y_{k+1})$, where $x_0 = y_0 = 0$ and $x_{K+1} = y_{K+1} = 1$. The market-clearing price in state ω must be such that the mass of agents bidding strictly above that price is no more than $1 - \alpha$, the number of units of the good, and such that the mass of agents bidding strictly less than that price is no more than α. Hence, the price p_k is a potential market-clearing price in state $\omega \in [0, 1]$ if

$$(2.45)\qquad \alpha F(x_k|\omega) + (1 - \alpha)F(y_k|\omega) \le \alpha \le \alpha F(x_{k+1}|\omega) + (1 - \alpha)F(y_{k+1}|\omega).$$

Note that because $x_0 = y_0 = 0$ and $x_{K+1} = y_{K+1} = 1$, (2.45) holds for at least one $k = 0, 1, \ldots, K$. However, if $\mathbf{x}$ and $\mathbf{y}$ yield buyer–seller bid functions in which there is no trade, there may be many prices p_k that satisfy (2.45). This happens only when, for some k, one of x_k and y_k is zero and the other is one. Such vectors can be avoided by restricting attention to $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon$ for $\varepsilon \in [0, 1)$. In this case, for all but perhaps finitely many $\omega \in [0, 1]$, there is a unique p_k that satisfies (2.45). We state this formally:

LEMMA 2.14: *If* $\varepsilon \in [0, 1)$ *and* $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon$, *then for all but perhaps finitely many* $\omega \in [0, 1]$, *there exists a unique* $k = 0, 1, \ldots, K$ *such that*

$$(2.46)\qquad \alpha F(x_k|\omega) + (1 - \alpha)F(y_k|\omega) < \alpha < \alpha F(x_{k+1}|\omega) + (1 - \alpha)F(y_{k+1}|\omega).$$

PROOF: Given strict affiliation, we have $F_\omega(x|\omega) < 0$ for all $x \in (0, 1)$. Consequently, if

$$(2.47)\qquad \alpha F(x_k|\omega) + (1 - \alpha)F(y_k|\omega) = \alpha$$

for some $k = 0, 1, \ldots, K + 1$, then, because $\alpha \in (0, 1)$ and $|x_k - y_k| \le \varepsilon < 1$, either x_k or y_k must be strictly between zero and one. However, this means that the left-hand side of (2.47) is strictly decreasing in ω and so, for that k, the equality can hold for exactly one value of ω.

Hence, for all but finitely many ω, when (2.45) holds for some k (and it must hold for at least one k), both inequalities are strict, i.e., (2.46) holds. This clearly implies that it holds for precisely one value of k. *Q.E.D.*

So, when $\varepsilon \in [0, 1)$ and $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon$, the price function $P(\cdot)$ determined by the market-clearing condition (2.45) is essentially uniquely determined. Note also that it is nondecreasing. At each of the finitely many $\omega \in [0, 1]$ where $P(\omega)$ is not uniquely determined, we may define $P(\omega)$ so that it is right-continuous at ω and continuous at ω if $\omega = 1$. The unique price function $P: [0, 1] \to \mathcal{P}$ determined in this way is said to be that *induced* by $(\mathbf{x}, \mathbf{y})$.

Assume, for the remainder of this part of the proof, that $\varepsilon \in [0, 1)$ so that the market-clearing price induced by any $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon$ is well defined.

When $P(\omega) = p_k$ and the buyer bids p_k, he will typically be rationed with positive probability. All agents who bid strictly more than p_k end up with a unit of the good. The mass of such agents is $\alpha(1 - F(x_{k+1}|\omega)) + (1 - \alpha)(1 - F(y_{k+1}|\omega))$, which is no more than $1 - \alpha$, the number of units of the good. The leftover units are randomly (uniformly) allocated to those agents who bid p_k, whose mass is $\alpha(F(x_{k+1}|\omega) - F(x_k|\omega)) + (1 - \alpha)(F(y_{k+1}|\omega) - F(y_k|\omega))$. Hence, the probability that an agent bidding p_k ends up with a unit is

$$\text{(2.48)} \quad \lambda(\omega|p_k, \mathbf{x}, \mathbf{y}) = \frac{\alpha F(x_{k+1}|\omega) + (1-\alpha)F(y_{k+1}|\omega) - \alpha}{\alpha(F(x_{k+1}|\omega) - F(x_k|\omega)) + (1-\alpha)(F(y_{k+1}|\omega) - F(y_k|\omega))}.$$

Given $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon$, let $\Psi_\varepsilon(\mathbf{x}, \mathbf{y})$ denote the set of $(\mathbf{x}', \mathbf{y}') \in C_\varepsilon$ such $\mathbf{x}'$ and $\mathbf{y}'$ each solve the ex ante maximization problem

$$\text{(2.49)} \quad \max_{\mathbf{z} \in X_K} \int_0^1 u(b_{\mathbf{z}}(x), x|\mathbf{x}, \mathbf{y}) f(x)\, dx.$$

As we shall demonstrate next, the integral in (2.49) is continuous in $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ so that $\Psi_\varepsilon(\mathbf{x}, \mathbf{y})$ is upper hemicontinuous and nonempty-valued. The latter follows because $(\mathbf{z}^0, \mathbf{z}^0) \in \Psi_\varepsilon(\mathbf{x}, \mathbf{y})$ for any solution $\mathbf{z}^0$ to (2.49). Let us now demonstrate the continuity of the integral in $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in C_\varepsilon \times X_K$.

LEMMA 2.15: *The integral* $\int_0^1 u(b_{\mathbf{z}}(x), x|\mathbf{x}, \mathbf{y}) f(x)\, dx$ *is continuous in* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *on* $C_\varepsilon \times X_K$.

PROOF: Suppose that $(\mathbf{x}^n, \mathbf{y}^n, \mathbf{z}^n) \to (\mathbf{x}, \mathbf{y}, \mathbf{z})$. Let $P(\cdot)$ denote the price function induced by $(\mathbf{x}, \mathbf{y})$ and, for each n, let $P_n(\cdot)$ denote the price function induced by $(\mathbf{x}^n, \mathbf{y}^n)$. Then

$$(2.50)\quad \int_0^1 u(b_{\mathbf{z}^n}(x), x|\mathbf{x}^n, \mathbf{y}^n)f(x)\,dx$$
$$= \int_0^1 \left[\int_{\omega:P_n(\omega)<b_{\mathbf{z}^n}(x)} (v(x,\omega) - P_n(\omega))h(\omega|x)\,d\omega\right] f(x)\,dx$$
$$+ \int_0^1 \left[\int_{\omega:P_n(\omega)=b_{\mathbf{z}^n}(x)} (v(x,\omega) - b_{\mathbf{z}^n}(x))\lambda(\omega|P_n(\omega), \mathbf{x}^n, \mathbf{y}^n) \times h(\omega|x)\,d\omega\right] f(x)\,dx.$$

Note first that for every $x \in [0,1]\backslash\{z_1, \dots, z_K\}$, $b_{\mathbf{z}^n}(x) = b_{\mathbf{z}}(x)$ for all n large enough. Consequently, if for almost every $\omega \in [0,1]$, $P_n(\omega) = P(\omega)$ for all n large enough, then by Lebesgue's dominated convergence theorem, the right-hand side of (2.50) converges to the same expression without the subscript or superscript n's. That is, it converges to $\int_0^1 u(b_{\mathbf{z}}(x), x|\mathbf{x}, \mathbf{y})f(x)\,dx$, as desired. Hence, it remains only to establish that for almost every $\omega \in [0,1]$, $P_n(\omega) = P(\omega)$ for all n large enough. We will in fact establish slightly more.

Recall that by Lemma 2.14, because $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon$, (2.46) holds for some k for all but perhaps finitely many $\omega \in [0,1]$. Let ω be any such state and let p_k be the unique price such that (2.46) holds for k. Consequently, $P(\omega) = p_k$. However, (2.46) clearly holds for the same k when $(\mathbf{x}, \mathbf{y})$ is replaced by $(\mathbf{x}^n, \mathbf{y}^n)$ and n is large enough. Hence, $P_n(\omega) = p_k$ for all n large enough. Because ω was arbitrary, we have established that for all but perhaps finitely many $\omega \in [0,1]$, $P_n(\omega) = P(\omega)$ for all n large enough. *Q.E.D.*

Thus, $\Psi_\varepsilon(\cdot,\cdot)$ is a nonempty-valued, compact-valued, upper hemicontinuous correspondence from C_ε into subsets of itself. However, it need not be convex-valued. Letting $\mathrm{co}\,\Psi_\varepsilon(\mathbf{x}, \mathbf{y})$ denote the convex hull of $\Psi_\varepsilon(\mathbf{x}, \mathbf{y})$, it follows from Kakutani's theorem that $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \mathrm{co}\,\Psi_\varepsilon(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ for some $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in C_\varepsilon$. We now establish two important results concerning the fixed points of $\mathrm{co}\,\Psi_\varepsilon(\cdot,\cdot)$.

LEMMA 2.16: *For every $\bar{\eta} > 0$ and every $\hat{\varepsilon} \in [0,1)$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$, all $\varepsilon \in [0, \hat{\varepsilon}]$, and all $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathrm{co}\,\Psi_\varepsilon(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, the length of each interval over which the $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$-induced price function $P(\cdot)$ is constant is strictly less than $\bar{\eta}$.*

PROOF: The proof is virtually identical to that of Lemma 2.2, whose steps can be followed *mutatis mutandis*, with the following two observations. First, consider arbitrary sequences $\varepsilon_r \in [0,1)$ and $(\mathbf{x}^r, \mathbf{y}^r) \in \mathrm{co}\,\Psi_{\varepsilon^r}(\mathbf{x}^r, \mathbf{y}^r)$, and the induced sequence of price functions $P_r(\cdot)$. Fix some price p_k in the grid $\bar{\mathcal{P}}^\Delta$ (the grid is independent of r). If the length of the interval of states on which P_r is p_k

is bounded away from zero along the sequence, then one of the differences $x^r_{k+1} - x^r_k$ or $y^r_{k+1} - y^r_k$ is also bounded away from zero. This follows from (2.45) and the strict MLRP.

Second, the rationing probability $\lambda(\omega|p_k, \mathbf{x}, \mathbf{y})$ in (2.48) is strictly decreasing in ω unless $x_k = y_k = 0$ and $x_{k+1} = y_{k+1} = 1$.[17] Consequently, because $(\mathbf{x}^r, \mathbf{y}^r) \in C_{\bar{\varepsilon}}$ and $\bar{\varepsilon} < 1$, the pointwise limit of $\lambda(\cdot|p_k, \mathbf{x}^r, \mathbf{y}^r)$ is strictly decreasing on the relevant interval of states. *Q.E.D.*

Recall from RP Section 3 that $x(\omega)$ is the αth percentile of $F(\cdot|\omega)$. Let us call two bidding functions, $b(\cdot)$ and $s(\cdot)$, *outcome-equivalent* if $b(x(\omega)) = s(x(\omega))$ for every $\omega \in [0, 1)$. Hence, outcome-equivalent bidding functions coincide at signals between $x(0)$ and $x(1)$. We will also say that $\mathbf{x}, \mathbf{y} \in X_K$ are outcome-equivalent if $b_{\mathbf{x}}(\cdot)$ and $b_{\mathbf{y}}(\cdot)$ are outcome-equivalent.

The next lemma states that for all $\varepsilon \in [0, 1)$ and for all sufficiently small Δ, all fixed points of co $\Psi_\varepsilon(\cdot, \cdot)$ are pairs of outcome-equivalent vectors, each of which is a fixed point of $B(\cdot)$. Loosely speaking, if the price grid is sufficiently fine, the only equilibria of the continuum economy in which buyers and sellers use distinct strategies are equilibria with arbitrarily little trade.

LEMMA 2.17: *For every $\hat{\varepsilon} \in [0, 1)$, there exists $\bar{\Delta} > 0$ such that for all $\Delta < \bar{\Delta}$ and all $\varepsilon \in [0, \hat{\varepsilon}]$, if $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \operatorname{co} \Psi_\varepsilon(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, then $\bar{\mathbf{x}} \in B(\bar{\mathbf{x}})$, $\bar{\mathbf{y}} \in B(\bar{\mathbf{y}})$, and $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are outcome-equivalent.*

PROOF: Choose $\bar{\eta} > 0$ as in Lemma 2.5. Given $\bar{\eta}$ and $\hat{\varepsilon} \in [0, 1)$, choose $\bar{\Delta}$ according to Lemma 2.16 and also so that the conclusion of Lemma 2.3 holds. Fix any $\Delta \in (0, \bar{\Delta})$, any $\varepsilon \in [0, \hat{\varepsilon}]$, and any $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \operatorname{co} \Psi_\varepsilon(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Because $u(p, x|\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is constant in $p \in \bar{\mathcal{P}}^\Delta$ for both p below $P(0)$ and also for p above $P(1)$, Lemmas 2.16 and 2.5 together imply that $u(p, x|\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies single crossing on $[0, 1] \times \bar{\mathcal{P}}^\Delta$. Consequently, $u(p, x|\bar{\mathbf{x}}, \bar{\mathbf{y}})$ has a nondecreasing pointwise maximizer and $\Psi_\varepsilon(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is convex so that $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \Psi_\varepsilon(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, which together imply that both $b_{\bar{\mathbf{x}}}(x)$ and $b_{\bar{\mathbf{y}}}(x)$ maximize $u(p, x|\bar{\mathbf{x}}, \bar{\mathbf{y}})$ over $p \in \bar{\mathcal{P}}^\Delta$ for every $x \in [0, 1]$. Now, because Lemma 2.5 yields strict single crossing for all price pairs that contain a price in the range of $P(\cdot)$ weakly between them, when a pointwise maximizer of $u(p, x|\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is weakly between $P(0)$ and $P(1)$, it is unique except for finitely many $x \in [0, 1]$. Hence, because $b_{\bar{\mathbf{x}}}(\cdot)$ and $b_{\bar{\mathbf{y}}}(\cdot)$ are right-continuous, for all p weakly between $P(0)$ and $P(1)$, $b^{-1}_{\bar{\mathbf{x}}}(p) = b^{-1}_{\bar{\mathbf{y}}}(p)$, so that $b_{\bar{\mathbf{x}}}(x(\omega)) = b_{\bar{\mathbf{y}}}(x(\omega))$ for all $\omega \in [0, 1)$. However, then $P(\omega) = b_{\bar{\mathbf{x}}}(x(\omega)) = b_{\bar{\mathbf{y}}}(x(\omega))$ for all $\omega \in [0, 1)$. Consequently, $b_{\bar{\mathbf{x}}}(\cdot)$ and $b_{\bar{\mathbf{y}}}(\cdot)$ are outcome-equivalent and both are symmetric equilibria. The latter implies $\bar{\mathbf{x}} \in B(\bar{\mathbf{x}})$ and $\bar{\mathbf{y}} \in B(\bar{\mathbf{y}})$. *Q.E.D.*

[17] To see this, let $a(\omega) = \alpha F(x_k|\omega) + (1-\alpha)F(y_k|\omega)$ and $b(\omega) = \alpha F(x_{k+1}|\omega) + (1-\alpha)F(y_{k+1}|\omega)$, and note that $\lambda(\omega|p) = (b(\omega) - \alpha)/[(b(\omega) - \alpha) + (\alpha - a(\omega))]$, that all terms in parentheses are nonnegative, and that $a'(\omega) \le 0$, $b'(\omega) \le 0$.

2.4. *Part D*

Consider the finite economy $\mathcal{E}(n, m, v, f, g, \Delta)$ with n buyers, m sellers, and price grid $\mathcal{P} = \{0, \Delta, 2\Delta, \ldots\}$. We wish to argue that even though one's bid can effect the price here, for the purposes of equilibrium existence, it remains without loss to restrict attention to the price grid $\bar{\mathcal{P}} = \{0, \Delta, 2\Delta, \ldots, K\Delta\}$, where $(K-1)\Delta < v(1,1) \leq K\Delta$. Indeed, suppose that $m > 1$ and all bidders other than i submit bids weakly below $K\Delta$. Let p denote the ex post market-clearing price if bidder i were to submit the bid $K\Delta$. Clearly, $p \leq K\Delta$. If instead i bids $b' > K\Delta$, the price must weakly increase to p' (by the auction rules; see RP Section 4), but it must remain weakly below $K\Delta$ (otherwise there would be excess supply because $m > 1$). Hence b' is guaranteed to win a unit at price $p' \geq p$. Therefore, b' is strictly worse than bidding $K\Delta$ if $p < p' \leq K\Delta$, because both bids are sure to win, but b' wins at a higher price, and b' is no better than $K\Delta$ if $p = p' = K\Delta$ because b' wins for sure and $K\Delta \geq v(1,1)$.

Fix $\alpha \in (0,1)$ for the remainder of the proof. We wish to establish our main result, namely, that for a residual set of value functions $v \in V$, there exists $\bar{\Delta} > 0$ such that for a residual set of $\Delta \in (0, \bar{\Delta})$ and all unbounded sequences of natural numbers $\{n_r\}, \{m_r\}$ such that $n_r/(n_r + m_r) \to_r \alpha$, the economy $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$ possesses a double-auction equilibrium for all r sufficiently large.

For $\mathbf{x}, \mathbf{y} \in X_K$, let $u_r^\beta(p, x|\mathbf{x}, \mathbf{y})$ denote the double-auction expected payoff of a buyer in $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$ whose signal is x when he bids $p \in \bar{\mathcal{P}}$, all other $n_r - 1$ buyers employ the bidding function $b_{\mathbf{x}}(\cdot)$, and all m_r sellers employ the bidding function $b_{\mathbf{y}}(\cdot)$. Similarly, let $u_r^\sigma(p, x|\mathbf{x}, \mathbf{y})$ denote the double-auction expected payoff of a seller whose signal is x when he bids $p \in \bar{\mathcal{P}}$, all n_r buyers employ $b_{\mathbf{x}}(\cdot)$, and all other $m_r - 1$ sellers employ $b_{\mathbf{y}}(\cdot)$.

Formally, the buyer's payoff $u_r^\beta(p, x|\mathbf{x}, \mathbf{y})$ is

$$(2.51) \quad \int_0^1 \sum_{\mathbf{B}^r : \rho^r(\mathbf{B}^r, p) < p} (v(x, \omega) - \rho^r(\mathbf{B}^r, p)) \Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega) h(\omega|x)\, d\omega$$
$$+ \int_0^1 \sum_{\mathbf{B}^r : \rho^r(\mathbf{B}^r, p) = p} (v(x, \omega) - \rho^r(\mathbf{B}^r, p)) \lambda^r(\mathbf{B}^r, p) \Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega)$$
$$\times h(\omega|x)\, d\omega,$$

where $\mathbf{B}^r = (B_1^r, \ldots, B_{n_r+m_r-1}^r)$ is the vector of all *others*' bids, from highest B_1^r to lowest $B_{n_r+m_r-1}^r$; $\rho^r(\mathbf{B}^r, p)$ is the market-clearing price that lies weakly between the m_rth highest and (m_r+1)st highest among *all* $n_r + m_r$ bids[18] $p, B_1^r, \ldots, B_{n_r+m_r-1}^r$; $\Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega)$ is the probability that the others' ordered

[18]The values taken on by $\rho^r(\mathbf{B}^r, p)$ are not required to lie in $\mathcal{P}$, unless, of course, the mth and $(m+1)$st highest bids are identical, in which case $\rho^r(\mathbf{B}^r, p)$ must be equal to that bid.

vector of bids is $\mathbf{B}^r$ conditional on the state ω and conditional on the others' strategies $(\mathbf{x}, \mathbf{y})$; and $\lambda^r(\mathbf{B}^r, p)$ is the probability the buyer receives a unit when his bid is p and the vector of the others' bids is $\mathbf{B}^r$. Formally,

$$\lambda^r(\mathbf{B}^r, p) = \begin{cases} 0, & \text{if } \rho^r(\mathbf{B}^r, p) > p, \\ \frac{m_r - \#\{\text{bids} > p\}}{\#\{\text{bids} = p\}}, & \text{if } \rho^r(\mathbf{B}^r, p) = p, \\ 1, & \text{if } \rho^r(\mathbf{B}^r, p) < p, \end{cases}$$

where the term "bids" refers to *all* bids, including p, among the $n_r + m_r$ bids in the vector $(\mathbf{B}^r, p)$.

When necessary, we will let $\tilde{\mathbf{B}}^r = (\tilde{B}^r_1, \dots, \tilde{B}^r_{n_r+m_r-1})$ denote the random vector that takes on the value $\mathbf{B}^r$ with probability $\Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega)$, given $\mathbf{x}, \mathbf{y}$, and ω.

Now, strictly speaking, the seller's payoff is

$$\begin{aligned} &\int_0^1 \sum_{\mathbf{B}^r : \rho^r(\mathbf{B}^r, p) > p} (\rho^r(\mathbf{B}^r, p) - v(x, \omega)) \Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega) h(\omega|x)\, d\omega \\ &\quad + \int_0^1 \sum_{\mathbf{B}^r : \rho^r(\mathbf{B}^r, p) = p} (\rho^r(\mathbf{B}^r, p) - v(x, \omega))(1 - \lambda^r(\mathbf{B}^r, p)) \Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega) \\ &\qquad \times h(\omega|x)\, d\omega. \end{aligned}$$

It will be convenient for the statements and proofs of several results (in particular, Lemmas 2.18 and 2.21) to define the seller's payoff as the preceding expression plus a function that is independent of p, the seller's choice variable. This of course has no effect on the seller's best replies and hence no effect on the set of equilibria.[19] Formally then, define the seller's payoff, $u^\sigma_r(p, x|\mathbf{x}, \mathbf{y})$ to be

$$\begin{aligned} (2.52) \quad &\int_0^1 \sum_{\mathbf{B}^r : \rho^r(\mathbf{B}^r, p) > p} (\rho^r(\mathbf{B}^r, p) - v(x, \omega)) \Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega) h(\omega|x)\, d\omega \\ &\quad + \int_0^1 \sum_{\mathbf{B}^r : \rho^r(\mathbf{B}^r, p) = p} (\rho^r(\mathbf{B}^r, p) - v(x, \omega))(1 - \lambda^r(\mathbf{B}^r, p)) \Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega) \\ &\qquad \times h(\omega|x)\, d\omega \\ &\quad + \int_0^1 \sum_{\mathbf{B}^r} (B^r_{m_r} - v(x, \omega)) \Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega) h(\omega|x)\, d\omega. \end{aligned}$$

[19] A similar device was used in Part A when we noted there, because of the continuum of agents, that the addition of a suitable constant to the seller's payoff function rendered it identical to the buyer's.

Note that we are abusing notation here because the random variable $\tilde{\mathbf{B}}^r$ from a seller's perspective is *not* the same as $\tilde{\mathbf{B}}^r$ from the buyer's perspective. For each agent, $\mathbf{B}^r$ is the ordered vector of bids of all *other* agents. Consequently, $\Pr(\mathbf{B}^r|\mathbf{x}, \mathbf{y}, \omega)$ is different in the two expressions because a buyer faces $n_r - 1$ buyers and m_r sellers, whereas a seller faces n_r buyers and $m_r - 1$ sellers. However, there is no need to introduce additional notation for this distinction because we will henceforth not explicitly employ the formula for the seller's payoff; our analysis of the seller will follow by analogy from the results that we prove for the buyer.

For $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in X_K$, the pair of functions $(b_{\hat{\mathbf{x}}}(\cdot), b_{\hat{\mathbf{y}}}(\cdot))$, each mapping $[0, 1]$ into $\bar{\mathcal{P}}$, constitutes a *double-auction equilibrium for* $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$ if for every $x \in [0, 1]$,

$$b_{\hat{\mathbf{x}}}(x) \text{ solves } \max_{p\in\bar{\mathcal{P}}} u_r^{\beta}(p, x|\hat{\mathbf{x}}, \hat{\mathbf{y}})$$

and

$$b_{\hat{\mathbf{y}}}(x) \text{ solves } \max_{p\in\bar{\mathcal{P}}} u_r^{\sigma}(p, x|\hat{\mathbf{x}}, \hat{\mathbf{y}}).$$

A double-auction equilibrium $(b_{\hat{\mathbf{x}}}(\cdot), b_{\hat{\mathbf{y}}}(\cdot))$ is called *nontrivial* if trade occurs with positive probability. That is, if for every $k = 1, \dots, K$, $(\hat{x}_k, \hat{y}_k) \neq (1, 0)$.

Recall that $u(p, x|\mathbf{x}, \mathbf{y})$, given by (2.44), is the double-auction expected payoff of an agent i (buyer or seller) in the continuum economy $\mathcal{E}(\alpha, v, f, g, \Delta)$ in which a fraction α of the agents are buyers, when agent i's signal is x and he bids p, and when, except for agent i, all buyers employ $b_{\mathbf{x}}(\cdot)$ and all sellers employ $b_{\mathbf{y}}(\cdot)$. We next show that an agent's finite economy payoff converges uniformly to his continuum economy payoff.

For the remainder of the proof, it is assumed that $n_r/(n_r + m_r) \to_r \alpha$.

LEMMA 2.18: *If* $\varepsilon \in [0, 1)$, *then as* $r \to \infty$,

$$u_r^{\beta}(p, x|\mathbf{x}, \mathbf{y}) \textit{ and } u_r^{\sigma}(p, x|\mathbf{x}, \mathbf{y}) \textit{ converge to } u(p, x|\mathbf{x}, \mathbf{y})$$

and

$$\frac{\partial}{\partial x} u_r^{\beta}(p, x|\mathbf{x}, \mathbf{y}) \quad \textit{and} \quad \frac{\partial}{\partial x} u_r^{\sigma}(p, x|\mathbf{x}, \mathbf{y})$$
$$\textit{converge to} \quad \frac{\partial}{\partial x} u(p, x|\mathbf{x}, \mathbf{y}),$$

where in each case the convergence is uniform in $p \in \bar{\mathcal{P}}$, $x \in [0, 1]$, *and* $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon$.

PROOF: We consider the buyer only; the proof for the seller is similar. Because $\bar{\mathcal{P}}$ is finite and both $[0, 1]$ and C_ε are compact, it suffices to consider a fixed price $p \in \bar{\mathcal{P}}$, and sequences $x^r \to x$ and $(\mathbf{x}^r, \mathbf{y}^r) \to (\mathbf{x}, \mathbf{y})$, and to show that

$$\lim_r u_r^\beta(p, x^r | \mathbf{x}^r, \mathbf{y}^r) = u(p, x | \mathbf{x}, \mathbf{y}). \tag{2.53}$$

Note that if $B^r_{m_r-1} = B^r_{m_r} = B^r_{m_r+1}$, then the market-clearing price $\rho^r(\mathbf{B}^r, p)$ is independent of the buyer's bid p and is equal to $B^r_{m_r}$. Let E^r denote the set of the others' ordered bids $(B^r_1, \dots, B^r_{n_r+m_r-1})$ such that $B^r_{m_r-1} = B^r_{m_r} = B^r_{m_r+1}$. Then we may write the buyer's payoff (2.51) as

$$\begin{aligned} &u_r^\beta(p, x^r | \mathbf{x}^r, \mathbf{y}^r) \\ &\quad = \int_0^1 \Bigg(\sum_{\mathbf{B}^r \in E^r : B^r_{m_r} < p} (v(x^r, \omega) - B^r_{m_r}) \Pr(\mathbf{B}^r | \mathbf{x}^r, \mathbf{y}^r, \omega) h(\omega | x^r) \Bigg) \, d\omega \\ &\qquad + \int_0^1 \Bigg(\sum_{\mathbf{B}^r \in E^r : B^r_{m_r} = p} (v(x^r, \omega) - B^r_{m_r}) \\ &\qquad\qquad \times \lambda^r(\mathbf{B}^r, p) \Pr(\mathbf{B}^r | \mathbf{x}^r, \mathbf{y}^r, \omega) h(\omega | x^r) \Bigg) \, d\omega \\ &\qquad + \int_0^1 \Bigg(\sum_{\mathbf{B}^r \notin E^r : \rho^r(\mathbf{B}^r, p) < p} (v(x^r, \omega) - \rho^r(\mathbf{B}^r, p)) \\ &\qquad\qquad \times \Pr(\mathbf{B}^r | \mathbf{x}^r, \mathbf{y}^r, \omega) h(\omega | x^r) \Bigg) \, d\omega \\ &\qquad + \int_0^1 \Bigg(\sum_{\mathbf{B}^r \notin E^r : \rho^r(\mathbf{B}^r, p) = p} (v(x^r, \omega) - \rho^r(\mathbf{B}^r, p)) \\ &\qquad\qquad \times \lambda^r(\mathbf{B}^r, p) \Pr(\mathbf{B}^r | \mathbf{x}^r, \mathbf{y}^r, \omega) h(\omega | x^r) \Bigg) \, d\omega. \end{aligned} \tag{2.54}$$

Let $P(\cdot)$ be the price function induced by $(\mathbf{x}, \mathbf{y})$ in $\mathcal{E}(\alpha, v, f, g, \Delta)$. Then by Lemmas 2.19 and 2.20 (see subsequent text),

$$\lim_r \Pr\big(\tilde{\mathbf{B}}^r \in E^r \text{ and } \tilde{B}^r_{m_r} = P(\omega) | \mathbf{x}^r, \mathbf{y}^r, \omega\big) = 1 \tag{2.55}$$

and

$$\lim_r \sum_{\mathbf{B}^r \in E^r : B^r_{m_r} = P(\omega)} \lambda^r(\mathbf{B}^r, P(\omega)) \Pr(\mathbf{B}^r | \mathbf{x}^r, \mathbf{y}^r, \omega) = \lambda(\omega | P(\omega), \mathbf{x}, \mathbf{y}) \tag{2.56}$$

for all but finitely many $\omega \in [0, 1]$.

Consider now each of the four functions of ω that appear in parentheses in (2.54). By (2.55) and because $x^r \to x$, the function on the first line converges to

$$(v(x,\omega) - P(\omega))\mathbf{I}_{P(\omega)<p}(\omega)h(\omega|x),$$

and those on the third and fourth lines converge to zero. By (2.55), (2.56), and because $x^r \to x$, the function in parentheses on the second line converges to

$$(v(x,\omega) - P(\omega))\lambda(\omega|P(\omega),\mathbf{x},\mathbf{y})\mathbf{I}_{P(\omega)=p}(\omega)h(\omega|x),$$

where, in each case, convergence is pointwise in $\omega \in [0,1]$ except possibly at finitely many points. Hence, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\lim_r u_r^\beta(p, x^r|\mathbf{x}^r,\mathbf{y}^r)\\
&\quad = \int_{\omega:P(\omega)<p} (v(x,\omega) - P(\omega))h(\omega|x)\,d\omega\\
&\qquad + \int_{\omega:P(\omega)=p} (v(x,\omega) - P(\omega))\lambda(\omega|p,\mathbf{x},\mathbf{y})h(\omega|x)\,d\omega\\
&\quad = u(p, x|\mathbf{x},\mathbf{y}),
\end{aligned}$$

as desired.

The result for $\partial u_r^\beta(p, x^r|\mathbf{x}^r,\mathbf{y}^r)/\partial x$ follows similarly because this partial derivative is equal to

$$\begin{aligned}
&\int_0^1 \sum_{\mathbf{B}^r:\rho^r(\mathbf{B}^r,p)<p} v_x(x^r,\omega)\Pr(\mathbf{B}^r|\mathbf{x}^r,\mathbf{y}^r,\omega)h(\omega|x^r)\,d\omega\\
&\quad - \int_0^1 \sum_{\mathbf{B}^r:\rho^r(\mathbf{B}^r,p)<p} \rho^r(\mathbf{B}^r,p)\Pr(\mathbf{B}^r|\mathbf{x}^r,\mathbf{y}^r,\omega)h_x(\omega|x^r)\,d\omega\\
&+ \int_0^1 \sum_{\mathbf{B}^r:\rho^r(\mathbf{B}^r,p)=p} v_x(x^r,\omega)\lambda^r(\mathbf{B}^r,p)\Pr(\mathbf{B}^r|\mathbf{x}^r,\mathbf{y}^r,\omega)h(\omega|x^r)\,d\omega\\
&- \int_0^1 \sum_{\mathbf{B}^r:\rho^r(\mathbf{B}^r,p)=p} \rho^r(\mathbf{B}^r,p)\lambda^r(\mathbf{B}^r,p)\Pr(\mathbf{B}^r|\mathbf{x}^r,\mathbf{y}^r,\omega)h_x(\omega|x^r)\,d\omega
\end{aligned}$$

and, as before, this expression converges to

$$\begin{aligned}
&\int_{\omega:P(\omega)<p} v_x(x,\omega)h(\omega|x)\,d\omega\\
&\qquad - \int_{\omega:P(\omega)<p} P(\omega)h_x(\omega|x)
\end{aligned}$$

$$+\int_{\omega:P(\omega)=p} v_x(x,\omega)\lambda(\omega|p,\mathbf{x},\mathbf{y})h(\omega|x)\,d\omega$$
$$-\int_{\omega:P(\omega)=p} P(\omega)\lambda(\omega|p,\mathbf{x},\mathbf{y})h_x(\omega|x)\,d\omega$$
$$=\frac{\partial}{\partial x}u(p,x|\mathbf{x},\mathbf{y}),$$

as desired. *Q.E.D.*

We now state and prove the two lemmas referred to in the preceding proof.

LEMMA 2.19: *If* $(\mathbf{x}^r,\mathbf{y}^r)\in C_\varepsilon$ *for* $\varepsilon\in[0,1)$ *and every* r, *and* $(\mathbf{x}^r,\mathbf{y}^r)\to(\mathbf{x},\mathbf{y})$, *then for all but finitely many* $\omega\in[0,1]$,

$$\lim_r \Pr\big(\tilde{B}^r_{m_r-1}=\tilde{B}^r_{m_r}=\tilde{B}^r_{m_r+1}=P(\omega)|\mathbf{x}^r,\mathbf{y}^r,\omega\big)=1.$$

PROOF: It suffices to show that

$$\lim_r \Pr(\tilde{B}^r_{m_r-1}=P(\omega)|\mathbf{x}^r,\mathbf{y}^r,\omega)=1,$$

$$\lim_r \Pr(\tilde{B}^r_{m_r}=P(\omega)|\mathbf{x}^r,\mathbf{y}^r,\omega)=1, \tag{2.57}$$

and

$$\lim_r \Pr(\tilde{B}^r_{m_r+1}=P(\omega)|\mathbf{x}^r,\mathbf{y}^r,\omega)=1.$$

We show only (2.57); the proofs of the other two equalities are similar. Recall from the proof of Lemma 2.18 that $(\mathbf{x}^r,\mathbf{y}^r)\to(\mathbf{x},\mathbf{y})\in\mathbf{C}_\varepsilon$ and that $\varepsilon\in[0,1)$. Also recall that $P(\cdot)$ is the price function induced by $(\mathbf{x},\mathbf{y})$ in $\mathcal{E}(\alpha,v,f,g,\boldsymbol{\Delta})$. By Lemma 2.14, for all but finitely many $\omega\in[0,1]$, $P(\omega)=p_k$, where k satisfies

$$\alpha F(x_k|\omega)+(1-\alpha)F(y_k|\omega)<\alpha<\alpha F(x_{k+1}|\omega)+(1-\alpha)F(y_{k+1}|\omega). \tag{2.58}$$

Choosing such an $\omega\in[0,1]$, it suffices to show that (2.57) holds for this ω. The remainder of the argument is conditional on this ω.

Recall that $\tilde{B}^r_{m_r}$ is the m_rth highest bid among n_r+m_r-1 agents, where n_r-1 are buyers who each employ $\mathbf{x}^r$ and where m_r are sellers who each employ $\mathbf{y}^r$. Consequently, a buyer bids less than p_k when his signal is less than x^r_k and a seller bids less than p_k when his signal is less than y^r_k.

Let $\tilde{\theta}^r_\beta$ be the fraction of the n_r-1 buyers whose signals are less than x^r_k and let $\tilde{\theta}^r_\sigma$ be the fraction of the m_r sellers whose signals are less than y^r_k. Hence, when the $n_r - 1$ buyers employ $\mathbf{x}^r$ and the m_r sellers employ $\mathbf{y}^r$, the number of them who bid less than p_k is

$$(n_r - 1)\tilde{\theta}^r_\beta + m_r\tilde{\theta}^r_\sigma. \tag{2.59}$$

The mean and variance of $\tilde{\theta}^r_\beta$ are $F(x^r_k|\omega)$ and $F(x^r_k|\omega)(1 - F(x^r_k|\omega))/(n_r - 1)$, respectively, and the mean and variance of $\tilde{\theta}^r_\sigma$ are $F(y^r_k|\omega)$ and $F(y^r_k|\omega)(1 - F(y^r_k|\omega))/m_r$, respectively.[20] Hence, for every $\eta > 0$, Chebyshev's inequality yields

$$\lim_r \Pr\big(|\tilde{\theta}^r_\beta - F(x^r_k|\omega)| < \eta\big) = \lim_r \Pr\big(|\tilde{\theta}^r_\sigma - F(y^r_k|\omega)| < \eta\big) = 1.$$

Hence, because $x^r_k \to x_k$ and $y^r_k \to y_k$,

$$\lim_r \Pr\big(|\tilde{\theta}^r_\beta - F(x_k|\omega)| < \eta\big) = \lim_r \Pr\big(|\tilde{\theta}^r_\sigma - F(y_k|\omega)| < \eta\big) = 1,$$

which, for brevity, we write instead as

$$\mathrm{P}\lim_r \tilde{\theta}^r_\beta = F(x_k|\omega) \quad \text{and} \quad \mathrm{P}\lim_r \tilde{\theta}^r_\sigma = F(y_k|\omega).$$

Therefore, because $n_r/(n_r + m_r) \to \alpha$,

$$\begin{aligned}
\mathrm{P}\lim_r &\left(\frac{n_r - 1}{n_r + m_r - 1}\tilde{\theta}^r_\beta + \frac{m_r}{n_r + m_r - 1}\tilde{\theta}^r_\sigma\right) \\
&= \alpha F(x_k|\omega) + (1 - \alpha)F(y_k|\omega) \\
&< \alpha \\
&= \lim_r \frac{n_r - 1}{n_r + m_r - 1},
\end{aligned} \tag{2.60}$$

where the inequality follows from (2.58). Consequently,

$$\begin{aligned}
1 &= \lim_r \Pr\left(\frac{n_r - 1}{n_r + m_r - 1}\tilde{\theta}^r_\beta + \frac{m_r}{n_r + m_r - 1}\tilde{\theta}^r_\sigma < \frac{n_r - 1}{n_r + m_r - 1}\right) \\
&= \lim_r \Pr((n_r - 1)\tilde{\theta}^r_\beta + m_r\tilde{\theta}^r_\sigma < n_r - 1), \\
&= \lim_r \Pr(\#\{\text{bids strictly less than } p_k\} < n_r - 1),
\end{aligned}$$

[20] For example, note that $\tilde{\theta}^r_\beta$ is the average of $n_r - 1$ independent and identically distributed random variables each of whose value is zero if the corresponding buyer's signal is below x_k and is one otherwise.

which implies that

$$\lim_r \Pr(\tilde{B}^r_{m_r} \geq p_k | \omega, \mathbf{x}^r, \mathbf{y}^r) = 1.$$

By employing x^r_{k+1} and y^r_{k+1} to construct random variables that count the fractions of buyer and seller bids strictly above p_k, a similar argument establishes that

$$\lim_r \Pr(\tilde{B}^r_{m_r} \leq p_k | \omega, \mathbf{x}^r, \mathbf{y}^r) = 1,$$

which completes the proof. *Q.E.D.*

LEMMA 2.20: *If* $(\mathbf{x}^r, \mathbf{y}^r) \in C_\varepsilon$ *for* $\varepsilon \in [0, 1)$ *and every* r, *and* $(\mathbf{x}^r, \mathbf{y}^r) \to (\mathbf{x}, \mathbf{y})$, *then for all but finitely many* $\omega \in [0, 1]$,

$$\lim_r \sum_{\mathbf{B}^r \in E^r : B^r_{m_r} = P(\omega)} \lambda^r(\mathbf{B}^r, P(\omega)) \Pr(\mathbf{B}^r | \mathbf{x}^r, \mathbf{y}^r, \omega) = \lambda(\omega | P(\omega), \mathbf{x}, \mathbf{y}).$$

PROOF: As shown in the proof of Lemma 2.19, for all but finitely many $\omega \in [0, 1]$ there is a unique k that satisfies (2.46). Fix any such ω for the remainder of the proof.

Because, by Lemma 2.19,

$$\lim_r \Pr\big(\tilde{\mathbf{B}}^r \in E^r \text{ and } \tilde{B}^r_{m_r} = P(\omega) | \mathbf{x}^r, \mathbf{y}^r, \omega\big) = 1,$$

it suffices to show that

$$\lim_r \sum_{\mathbf{B}^r} \lambda^r(\mathbf{B}^r, P(\omega)) \Pr(\mathbf{B}^r | \mathbf{x}^r, \mathbf{y}^r, \omega) = \lambda(\omega | P(\omega), \mathbf{x}, \mathbf{y}),$$

and for this it suffices to show that

$$\operatorname{P}\lim_r \lambda^r(\tilde{\mathbf{B}}^r, P(\omega)) = \lambda(\omega | P(\omega), \mathbf{x}, \mathbf{y}).$$

Now, for each value $\mathbf{B}^r$ assumed by $\tilde{\mathbf{B}}^r$,

$$\lambda^r(\mathbf{B}^r, P(\omega)) = \begin{cases} 0, & \text{if } \rho^r(\mathbf{B}^r, P(\omega)) > P(\omega), \\ \frac{m_r - \#\{\text{bids} > P(\omega)\}}{\#\{\text{bids} = P(\omega)\}}, & \text{if } \rho^r(\mathbf{B}^r, P(\omega)) = P(\omega), \\ 1, & \text{if } \rho^r(\mathbf{B}^r, P(\omega)) < P(\omega). \end{cases}$$

Hence, because Lemma 2.19 implies that

$$\lim_r \Pr\big(\rho^r(\tilde{\mathbf{B}}^r, P(\omega)) = P(\omega) | \mathbf{x}^r, \mathbf{y}^r, \omega\big) = 1,$$

it suffices to show that

$$\operatorname{P}\lim_{r} \frac{m_r - \#\{\text{bids} > P(\omega)\}}{\#\{\text{bids} = P(\omega)\}} = \lambda(\omega|P(\omega), \mathbf{x}, \mathbf{y}),$$

which is equivalent to

$$\text{(2.61)} \qquad \frac{\lim_r \frac{m_r}{n_r+m_r} - \operatorname{P}\lim_r \frac{\#\{\text{bids}>P(\omega)\}}{n_r+m_r}}{\operatorname{P}\lim_r \frac{\#\{\text{bids}=P(\omega)\}}{n_r+m_r}} = \lambda(\omega|P(\omega), \mathbf{x}, \mathbf{y}),$$

as long as the denominator is not zero, and where all the agents' bids are considered, even the distinguished buyer whose bid is fixed and equal to $P(\omega)$.

Let us begin with the numerator of (2.61). The random quantity

$$\frac{\#\{\text{bids} > P(\omega)\}}{n_r + m_r}$$

is the fraction of bids among the random vector of bids $(\tilde{\mathbf{B}}^r, P(\omega))$ that are strictly greater than $P(\omega)$. In (2.60) we showed that among bids in the random vector $\tilde{\mathbf{B}}^r$, the fraction that are strictly *less* than $P(\omega) = p_k$ has probability limit equal to $\alpha F(x_k|\omega) + (1-\alpha)F(y_k|\omega)$. Because buyers bid strictly more than $P(\omega) = p_k$ when their signal is greater than x_{k+1} and sellers bid strictly more than $P(\omega) = p_k$ when their signal is greater than y_{k+1}, a similar argument establishes that

$$\operatorname{P}\lim_{r} \frac{\#\{\text{bids} > P(\omega)\}}{n_r + m_r} \\ = \alpha(1 - F(x_{k+1}|\omega)) + (1-\alpha)(1 - F(y_{k+1}|\omega));$$

together, the two results establish that

$$\operatorname{P}\lim_{r} \frac{\#\{\text{bids} = P(\omega)\}}{n_r + m_r} \\ = \alpha(F(x_{k+1}|\omega) - F(x_k|\omega)) + (1-\alpha)(F(y_{k+1}|\omega) - F(y_k|\omega)).$$

Hence, because $m_r/(n_r + m_r) \to 1 - \alpha$, the left-hand side of (2.61) becomes

$$\frac{\alpha F(x_{k+1}|\omega) + (1-\alpha)F(y_{k+1}|\omega) - \alpha}{\alpha(F(x_{k+1}|\omega) - F(x_k|\omega)) + (1-\alpha)(F(y_{k+1}|\omega) - F(y_k|\omega))},$$

which is precisely $\lambda(\omega|P(\omega), \mathbf{x}, \mathbf{y})$ (see(2.48)). Finally, note that the denominator is nonzero because (2.46) holds. *Q.E.D.*

As we have seen in RP Section 3, where bids can be any nonnegative real number, there is an indeterminacy in the equilibrium of the continuum economy. This remains true when bids are restricted to a grid. In particular, if $b(\cdot)$ is a double-auction equilibrium for the continuum economy $\mathcal{E}(\alpha, v, f, g, \Delta)$ with grid size Δ, there may be prices in the range of $b(\cdot)$ that never arise as market-clearing prices. Indeed, if $b(x) < b(x(0))$ or $b(x) > b(x(1))$, then $p = b(x)$ is such a price because $P(\omega) = b(x(\omega)) \in [b(x(0)), b(x(1))]$ for all $\omega \in [0, 1]$. Consequently, all bids below $P(0)$ are equally good and all bids above $P(1)$ are equally good, and changing an equilibrium bidding function outside the range of $P(\cdot)$ does not upset the equilibrium. Of course, all such equilibria are outcome-equivalent. This indeterminacy of equilibria in the continuum economy makes it difficult to pin down, in large finite economies, the interval of signals over which bidders bid prices that are outside the range of the limit price function $P(\cdot)$. In particular, the length of these intervals and, consequently, the probability that the associated prices occur, can vanish in the limit as the number of traders grows. This, in turn, makes it difficult to establish the single-crossing property for such prices. We overcome this by forcing bidders to submit bidding functions whose step widths are bounded away from zero for potentially problematic prices, a restriction that is ultimately not binding. We require some definitions.

Let $\mathcal{K}_0 = \{k \geq 0 : v(0, 0) - \Delta < k\Delta < v(x(0), 0)\}$ and let $\mathcal{K}_1 = \{k \geq 0 : v(x(1), 1) < k\Delta < v(1, 1) + \Delta\}$. The set $\mathcal{K}_0$ contains indices that correspond to low grid prices, at most one of which is below $v(0, 0)$, and $\mathcal{K}_1$ contains indices that correspond to high grid prices, at most one of which is above $v(1, 1)$. These "extreme" prices are those that can occur as equilibrium prices with vanishingly small probability in large finite economies.

For $\varepsilon \geq 0$, let $X_K^\varepsilon = \{\mathbf{x} \in X_K : x_{k+1} - x_k \geq \varepsilon \text{ for all } k \in \mathcal{K}_0 \cup \mathcal{K}_1\}$, where $x_0 = 0$ and $x_{K+1} = 1$. Hence, to be a member of X_K^ε a vector of jump points must induce a nondecreasing function such that, for every $k \in \mathcal{K}_0 \cup \mathcal{K}_1$, the length of the interval over which the function is $k\Delta$ is at least ε. Clearly, X_K^ε is compact and convex. A sufficient condition for nonemptiness is $\varepsilon \leq 1/(K + 1)$. We shall restrict bidders' vectors of jump points to be in the set X_K^ε.

We now define a correspondence for the finite economy $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$ whose fixed points will be shown to be double-auction equilibria when n_r and m_r are sufficiently large. Fix $\varepsilon \in [0, 1)$ and define $C_\varepsilon^0 = C_{\varepsilon^2} \cap (X_K^\varepsilon \times X_K^\varepsilon)$. Note the presence of the ε^2. This implies that when $\varepsilon \in (0, 1)$ and $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon^0$, we have $x_{k+1} - y_k = (x_{k+1} - x_k) + (x_k - y_k) \geq \varepsilon - \varepsilon^2 = \varepsilon(1 - \varepsilon) > 0$, for $k \in \mathcal{K}_0 \cup \mathcal{K}_1$. Similarly, $y_{k+1} - x_k > 0$.

The set C_ε^0 is nonempty whenever X_K^ε is nonempty. For each $(\mathbf{x}, \mathbf{y}) \in C_\varepsilon^0$, let $\Psi_\varepsilon^r(\mathbf{x}, \mathbf{y})$ denote the set of solutions to the ex ante maximization problem

$$\max_{(\mathbf{x}', \mathbf{y}') \in C_\varepsilon^0} \left[\int_0^1 u_r^\beta(b_{\mathbf{x}'}(x), x|\mathbf{x}, \mathbf{y}) f(x)\, dx + \int_0^1 u_r^\sigma(b_{\mathbf{y}'}(x), x|\mathbf{x}, \mathbf{y}) f(x)\, dx \right]. \tag{2.62}$$

As with the related maximization problems (2.5) and (2.49), the objective function here is jointly continuous in $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{y}$, and $\mathbf{y}'$. Hence, $\Psi_\varepsilon^r(\cdot, \cdot)$ is nonempty-valued when $X_K^\varepsilon \neq \emptyset$ and upper hemicontinuous, but it need not be convex-valued. Kakutani's theorem guarantees the existence of $(\mathbf{x}^r, \mathbf{y}^r) \in \operatorname{co}\Psi_\varepsilon^r(\mathbf{x}^r, \mathbf{y}^r)$.

We now connect the fixed points of the finite economy correspondence $\operatorname{co}\Psi_\varepsilon^r(\cdot,\cdot)$ with those of the continuum economy correspondence $\operatorname{co}\Psi_\varepsilon(\cdot,\cdot)$ from Part C.

LEMMA 2.21: *Suppose that* $\varepsilon \in [0,1)$ *and* $(\mathbf{x}^r, \mathbf{y}^r) \in \operatorname{co}\Psi_\varepsilon^r(\mathbf{x}^r, \mathbf{y}^r)$ *for* $r = 1, 2, \ldots$. *If* $(\mathbf{x}^r, \mathbf{y}^r) \to (\mathbf{x}, \mathbf{y})$, *then* $(\mathbf{x}, \mathbf{y}) \in \operatorname{co}\Psi_\varepsilon(\mathbf{x}, \mathbf{y})$.

PROOF: Note that $(\mathbf{x}^r, \mathbf{y}^r) \in \operatorname{co}\Psi_\varepsilon^r(\mathbf{x}^r, \mathbf{y}^r)$ implies that $\mathbf{x}^r, \mathbf{y}^r \in X_K^\varepsilon$ so that $X_K^\varepsilon \neq \emptyset$. By Caratheodory's theorem, for every r, $(\mathbf{x}^r, \mathbf{y}^r) \in \operatorname{co}\Psi_\varepsilon^r(\mathbf{x}^r, \mathbf{y}^r) \subseteq \mathbb{R}^{2K}$ can be expressed as a convex combination of $2K+1$ or fewer elements, $(\mathbf{x}^{r,1}, \mathbf{y}^{r,1}), \ldots, (\mathbf{x}^{r,2K+1}, \mathbf{y}^{r,2K+1})$, of $\Psi_\varepsilon^r(\mathbf{x}^r, \mathbf{y}^r)$. By Lemma 2.18, the finite economy payoffs of the buyers and sellers converge uniformly to their common payoff in the continuum economy. Hence, for each $j = 1, \ldots, 2K+1$, the limit $(\mathbf{x}^j, \mathbf{y}^j)$ of $(\mathbf{x}^{r,j}, \mathbf{y}^{r,j})$ must solve the maximization problem

$$\max_{(\mathbf{x}',\mathbf{y}')\in C_\varepsilon^0}\left[\int_0^1 u(b_{\mathbf{x}'}(x), x|\mathbf{x},\mathbf{y}) f(x)\,dx + \int_0^1 u(b_{\mathbf{y}'}(x), x|\mathbf{x},\mathbf{y}) f(x)\,dx\right]. \tag{2.63}$$

Let $\mathbf{z}^0 \in X_K^\varepsilon$ solve

$$\max_{\mathbf{z}\in \mathbf{X}_K^\varepsilon} \int_0^1 u(b_{\mathbf{z}}(x), x|\mathbf{x},\mathbf{y}) f(x)\,dx. \tag{2.64}$$

Then $(\mathbf{z}^0, \mathbf{z}^0) \in C_\varepsilon^0$ solves (2.63). Consequently, for each k, both $\mathbf{x}^j$ and $\mathbf{y}^j$ must solve (2.64), but because $C_{\varepsilon^2} \subseteq C_\varepsilon$, this implies (see (2.49)) that $(\mathbf{x}^j, \mathbf{y}^j) \in \Psi_\varepsilon(\mathbf{x}, \mathbf{y})$ for each j. Hence, because $(\mathbf{x}, \mathbf{y})$ is a convex combination of the $(\mathbf{x}^j, \mathbf{y}^j)$, we have $(\mathbf{x}, \mathbf{y}) \in \operatorname{co}\Psi_\varepsilon(\mathbf{x}, \mathbf{y})$, as desired. *Q.E.D.*

We next present a consequence of Assumption A.2 in RP that $f(x|\omega)$ satisfies the strict monotone likelihood ratio property.

LEMMA 2.22: *For all* $\omega_0 \in (0,1]$, *there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$, *if* $|x - y| \leq \varepsilon^2$ *and* $\varepsilon(1-\varepsilon) \leq x \leq 1/2$, *then* (a) $F(x|\omega_0/2) > F(y|\omega_0)$ *and* (b) $F(1-x|1-\omega_0) > F(1-y|1-\omega_0/2)$.

PROOF: We prove (a) only, because the proof of (b) is similar. Suppose that (a) fails. Then there exists $\omega_0 \in (0,1]$ and convergent sequences x_n, $y_n \in [0,1]$ and $\varepsilon_n \to 0$ such that $|x_n - y_n| \leq \varepsilon_n^2$, $\varepsilon_n(1-\varepsilon_n) \leq x_n \leq 1/2$, and $F(x_n|\omega_0/2) \leq F(y_n|\omega_0)$ for every n. Hence, x_n and y_n must converge to the same limit, $\hat{x}$, say, where $\hat{x} \leq 1/2$ and $F(\hat{x}|\omega_0/2) \leq F(\hat{x}|\omega_0)$. Therefore, $\hat{x} = 0$ because $F_\omega(x|\omega) < 0$ for $x \in (0,1)$. So, we have $x_n, y_n \to 0$.

For every n, $F(x_n|\omega_0/2) \leq F(y_n|\omega_0)$ implies $y_n > x_n$ because $F(x|\omega)$ is strictly increasing in x and strictly decreasing in ω. Therefore,

$$\begin{aligned} 0 &\leq \frac{F(y_n|\omega_0) - F(x_n|\omega_0/2)}{y_n - x_n} \\ &= \frac{F(y_n|\omega_0) - F(x_n|\omega_0) + F(x_n|\omega_0) - F(x_n|\omega_0/2)}{y_n - x_n} \\ &= \frac{F(y_n|\omega_0) - F(x_n|\omega_0)}{y_n - x_n} + \frac{F(x_n|\omega_0) - F(x_n|\omega_0/2)}{x_n} \frac{x_n}{y_n - x_n} \\ &\to -\infty, \end{aligned}$$

a contradiction, where the limit is justified as follows. The first term converges to $f(0|\omega_0)$ and so is bounded. As for the second term, $(F(x_n|\omega_0) - F(x_n|\omega_0/2))/x_n \to f(0|\omega_0) - f(0|\omega_0/2)$, while $x_n/(y_n - x_n) \geq \varepsilon_n(1-\varepsilon_n)/\varepsilon_n^2 \to +\infty$. Hence, the limit follows if $f(0|\omega_0) < f(0|\omega_0/2)$. Now, because $\omega_0 > 0$, strict MLRP implies that $f(x|\omega_0)/f(x|\omega_0/2)$ is strictly increasing in x. Hence, because both $f(\cdot|\omega_0)$ and $f(\cdot|\omega_0/2)$ are positive and integrate to unity over $[0, 1]$, neither can be almost everywhere above the other. Therefore, we must have $f(0|\omega_0) < f(0|\omega_0/2)$ as desired. *Q.E.D.*

LEMMA 2.23: *For every $\beta > 0$, there exists $\theta \in (0, 1)$ such that for all $a, b, c, d \geq 0$, if $0 \leq b - a \leq 1$ and $d \geq \max(b, c) + \beta$, then for all positive integers n and m,*

$$\frac{\int_a^b z^n(1-z)^m\,dz}{\int_c^d z^n(1-z)^m\,dz} \leq \frac{2}{\beta}\theta^{n+m}, \quad \textit{whenever} \quad \frac{n}{n+m} \geq d.$$

PROOF: Without loss, assume $a < b$ and let c' denote the average of $\max(b, c)$ and d. Hence, $d > c' > c$. If $d \leq n/(n+m)$, then

$$\begin{aligned} &\frac{\int_a^b z^n(1-z)^m\,dz}{\int_c^d z^n(1-z)^m\,dz} \\ &\quad < \frac{\int_a^b z^n(1-z)^m\,dz}{\int_{c'}^d z^n(1-z)^m\,dz} \\ &\quad = \frac{b-a}{d-c'} \frac{\frac{1}{b-a}\int_a^b [z^{n/(n+m)}(1-z)^{m/(n+m)}]^{n+m}\,dz}{\frac{1}{d-c'}\int_{c'}^d [z^{n/(n+m)}(1-z)^{m/(n+m)}]^{n+m}\,dz} \\ &\quad < \frac{1}{\beta/2}\left[\frac{b^{n/(n+m)}(1-b)^{m/(n+m)}}{(c')^{n/(n+m)}(1-c')^{m/(n+m)}}\right]^{n+m} \end{aligned}$$

$$\leq \frac{1}{\beta/2}\left[\frac{b^{n/(n+m)}(1-b)^{m/(n+m)}}{(b+\beta/2)^{n/(n+m)}(1-(b+\beta/2))^{m/(n+m)}}\right]^{n+m},$$

where the second strict inequality follows because $b-a \leq 1$, $d-c' \geq \beta/2$, and $z^{\gamma}(1-z)^{1-\gamma}$ is strictly increasing in z on $[0, \gamma]$ for all $\gamma \in (0, 1]$; note that $a < b < c' < d \leq n/(n+m)$. The fourth line again follows from the monotonicity of $z^{\gamma}(1-z)^{1-\gamma}$ and because $b+\beta/2 \leq c' \leq n/(n+m)$. Hence, considering the term in square brackets in the last line, it suffices to show that

$$\sup_{b,\gamma} \frac{b^{\gamma}(1-b)^{1-\gamma}}{(b+\beta/2)^{\gamma}(1-(b+\beta/2))^{1-\gamma}} < 1,$$

where, given $\beta > 0$, the supremum is taken over all $b, \gamma \geq 0$ such that $b + \beta/2 \leq \gamma \leq 1$. The desired result follows because, by the previously mentioned monotonicity property of $z^{\gamma}(1-z)^{1-\gamma}$, the ratio in the supremum can approach one only if γ approaches zero, which is impossible because $\gamma \geq \beta/2$. *Q.E.D.*

We state the following lemma without proof. It is a straightforward consequence of our assumptions that v and f are continuously differentiable on their domains. Recall that $h(\omega|x) = f(x|\omega)g(\omega)/\int f(x|\omega)g(\omega)\,d\omega$.

LEMMA 2.24: *Suppose a sequence of probability measures* $\{\mu_r\}$ *on* $[0, 1]$ *converges weakly to a mass point at* $\omega^* \in [0, 1]$. *For* $x \in [0, 1]$ *and each* r, *define*

$$\gamma_r(x) = \int_0^1 v(x, \omega)\frac{h(\omega|x)}{\int_0^1 h(\omega|x)\,d\mu_r(\omega)}\,d\mu_r(\omega).$$

Then $\gamma_r(x) \to_r v(x, \omega^*)$ *and* $\gamma_r'(x) \to_r v_x(x, \omega^*)$, *both uniformly in* x *on* $[0, 1]$.

Recall that $\omega(x)$ is the state ω in which the αth percentile of $F(\cdot|\omega)$ is closest to x. The following result proves (i) and (ii) of RP Theorem 6.1.

THEOREM 2.25: *Given* $\alpha \in (0, 1)$, *let* V^0 *denote the residual subset of* V *from Lemma* 2.13, *and suppose that* $n_r, m_r \to \infty$ *and* $n_r/(n_r + m_r) \to \alpha$. *For every* $v \in V^0$ *and every* $\eta_0 > 0$, *there exists* $\bar{\Delta} > 0$ *such that for a residual set of* $\Delta \in (0, \bar{\Delta})$, *the finite economy* $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$ *possesses a nontrivial double-auction equilibrium* $(b_{\mathbf{x}^r}(\cdot), b_{\mathbf{y}^r}(\cdot))$ *for all sufficiently large r. Furthermore,* $\lim_r b_{\mathbf{x}^r}(x) = \lim_r b_{\mathbf{y}^r}(x) = \hat{b}(x)$, *where the convergence is uniform in* $x \in [0, 1]$ *and where* $\hat{b}(\cdot)$ *is a double-auction equilibrium for the continuum economy* $\mathcal{E}(\alpha, v, f, g, \Delta)$, *such that* $\sup_{x\in[0,1]} |\hat{b}(x) - v(x, \omega(x))| < \bar{\eta}$.

PROOF: Fix any $\eta_0 > 0$, $\hat{\varepsilon} \in (0, 1)$, and any $v \in V^0$. Bacause V^0 is as in Lemma 2.13, $v(0, 0) > 0$. Choose $\bar{\eta} > 0$ as in Lemma 2.5. Choose $0 < \bar{\Delta} <$

$\min(\bar{\eta}, \eta_0)$ and a residual subset D of $(0, \bar{\Delta})$ on which the conclusions of Proposition 2.3 and Lemmas 2.8, 2.9, 2.13, 2.16, and 2.17 all hold, and such that for every $\Delta \in D$, none of $v(0,0)$, $v(1,1)$, $v(x(0),0)$, or $v(x(1),1)$ is an integer multiple of Δ. At the same time, choose $\bar{\Delta} < v(0,0)$ and such that for all $\Delta \in (0, \bar{\Delta})$ and all double-auction equilibria $b(\cdot)$ of $\mathcal{E}(\alpha, v, f, g, \Delta)$, we have $\sup_{x \in [x(0), x(1))} |b(x) - v(x, \omega(x))| < \eta_0$ (see Lemma 2.7).

Fix now any $\Delta \in D$, and choose K so that $(K-1)\Delta < v(1,1) \leq K\Delta$. This determines $\bar{\mathcal{P}} = \{0, \Delta, 2\Delta, \ldots, K\Delta\}$ and the correspondence $B(\cdot)$, as defined in Section 2.1. Given the choices of v, $\bar{\Delta}$, and Δ, we may choose $\bar{\varepsilon} > 0$ according to Lemma 2.8. We may also choose $\zeta_0 > 0$ such that $v_x(x, \omega) > \zeta_0$ for all x and ω.

Let $\underline{k} = \min \mathcal{K}_0$. Note that $\underline{k} \geq 1$ because $v(0,0) > \bar{\Delta} > \Delta$ and note that $K = \max \mathcal{K}_1$. Define $x^0_{\underline{k}} \equiv 0$ and, for each $\underline{k} < k \in \mathcal{K}_0$, define $x^0_k > 0$ so that $v(x^0_k, 0) = k\Delta$. Define $x^1_K \equiv 1$ and, for each $K > k \in \mathcal{K}_1$, define $x^1_k < 1$ so that $v(x^1_k, 1) = k\Delta$. Because v is strictly increasing in x, we may choose $\zeta_1 > 0$ such that (i) for $\omega = 0, 1$, $x^\omega_{k+1} - x^\omega_k > \zeta_1$ for all $k, k+1 \in \mathcal{K}_\omega$, and (ii) $x_{k_1} - x^0_{k_1 - 1} > \zeta_1$ and $x^1_{K_L} - x_{K_{L-1}+1} > \zeta_1$ for all $\mathbf{x} \in B(\mathbf{x})$, where $k_1\Delta < \cdots < k_{L-1}\Delta$ is the range of $P(\cdot) \equiv b_{\mathbf{x}}(x(\cdot))$. Note that (ii) can be satisfied by Lemma 2.8(a) and (b), and because $v_x > 0$ is bounded away from zero.

Choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \cdot \zeta_0 < \min(v(0,0) - \underline{k}\Delta, K\Delta - v(1,1), \zeta_1/2, 1)$, where the first two terms in the min are positive because neither $v(0,0)$ nor $v(1,1)$ is an integer multiple of Δ. By Lemma 2.24, we may choose $\omega_0 \in (0,1)$ such that for all sequences of probability measures, $\{\mu_r\}$ on $[0,1]$, if $\gamma_r(x)$ is defined as in Lemma 2.24, then for all r sufficiently large,

$$(2.65) \qquad |\gamma_r(x) - v(x,0)| + |\gamma'_r(x) - v_x(x,0)| < \zeta_0 \cdot \varepsilon_0 \quad \text{if} \quad \mu_r([0, \omega_0]) \to_r 1$$

and

$$(2.66) \qquad |\gamma_r(x) - v(x,1)| + |\gamma'_r(x) - v_x(x,1)| < \zeta_0 \cdot \varepsilon_0 \quad \text{if} \quad \mu_r([1-\omega_0, 1]) \to_r 1.$$

By Lemma 2.22, we may choose $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$, if $|x - y| \leq \varepsilon^2$ and $\varepsilon(1-\varepsilon) \leq x \leq 1/2$, then $F(x|\omega_0/2) > F(y|\omega_0)$ and $F(1-x|1-\omega_0) > F(1-y|1-\omega_0/2)$.

Finally, fix any strictly positive $\varepsilon < \min(\bar{\varepsilon}, \varepsilon_0, \varepsilon_1, \frac{1}{K+1})$. Together with our choice of Δ, this determines the correspondence $\Psi_\varepsilon(\cdot,\cdot)$ as defined in Section 2.3 and, given n_r and m_r for each r, also determines $\Psi^r_\varepsilon(\cdot,\cdot)$ as defined in Section 2.4.

Because $\varepsilon < 1/(K+1)$, $C_{\varepsilon^2} \cap (X^\varepsilon_K \times X^\varepsilon_K)$ is nonempty. Hence, for each $r = 1, 2, \ldots,$ Kakutani's theorem guarantees the existence of $(\mathbf{x}^r, \mathbf{y}^r) \in \operatorname{co} \Psi^r_\varepsilon(\mathbf{x}^r, \mathbf{y}^r)$, where $\Psi^r_\varepsilon(\cdot,\cdot)$ is defined as in (2.62) in the economy $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$. We shall first show that for all r sufficiently large,

$(b_{\mathbf{x}^r}(\cdot), b_{\mathbf{y}^r}(\cdot))$ constitutes a nontrivial double-auction equilibrium for $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$.

Without loss, we may suppose that $(\mathbf{x}^r, \mathbf{y}^r)$ converges to, say, $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. Consequently, $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in \Psi_\varepsilon(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ by Lemma 2.21, and so $\hat{\mathbf{x}} \in \operatorname{co} B(\hat{\mathbf{x}})$ and $\hat{\mathbf{y}} \in \operatorname{co} B(\hat{\mathbf{y}})$ are outcome-equivalent by Lemma 2.17. Letting $k_1\Delta < \cdots < k_{L-1}\Delta$ be the common range of $b_{\hat{\mathbf{x}}}(x(\cdot))$ and $b_{\hat{\mathbf{y}}}(x(\cdot))$, we therefore have $\hat{x}_{k_l} = \hat{y}_{k_l}$ for $l = 1, 2, \ldots, k_L$, where $k_L \equiv k_{L-1} + 1$. The choice of Δ and Lemma 2.9 yield

$$k \in \mathcal{K}_0 \quad \text{implies} \quad k \le k_1 \tag{2.67}$$

and

$$k \in \mathcal{K}_1 \quad \text{implies} \quad k \ge k_{L-1}. \tag{2.68}$$

For each $k \in \mathcal{K}_0 \backslash \{k_1\}$ define $x_k^* = x_k^0$, for each $k \in \mathcal{K}_1 \backslash \{k_L\}$ define $x_k^* = x_{k-1}^1$, and for $k \in \{k_1, \ldots, k_L\}$ define $x_k^* = \hat{x}_k$. Let $\mathcal{K} \equiv \mathcal{K}_0 \cup \{k_1, \ldots, k_L\} \cup \mathcal{K}_1$ and suppose that the following two conditions hold (we will establish them shortly):

(I) For all r large enough, $u_r^\beta(p, x|\mathbf{x}^r, \mathbf{y}^r)$ satisfies strict single crossing in (p, x) for all $x \in [0, 1]$ and all prices $p \in \bar{\mathcal{P}}$ that are, in the rth economy, best replies for the buyer for some signal $x \in [0, 1]$.[21]

(II) For all $k \in \mathcal{K}_0 \cup \mathcal{K}_1$, all $k' \neq k$, and all r large enough,

$$u_r^\beta(k\Delta, x|\mathbf{x}^r, \mathbf{y}^r) > u_r^\beta(k'\Delta, x|\mathbf{x}^r, \mathbf{y}^r), \quad \text{for all } x \in (x_k^* + \varepsilon, x_{k+1}^* - \varepsilon),$$

for all $x \in [0, x_{\underline{k}+1}^* - \varepsilon)$ when $k = \underline{k}$, and for all $x \in (x_K^* + \varepsilon, 1]$ when $k = K$.

Condition (I) implies that for all r large enough, there is a unique and nondecreasing function, $\bar{b}_r^\beta(\cdot)$ say, whose values solve for each $x \in [0, 1]$ the problem $\max u_r^\beta(p, x|\mathbf{x}^r, \mathbf{y}^r)$ over $p \in \bar{\mathcal{P}}$. Consequently, $\bar{b}_r^\beta(x) = k\Delta$ for all k and x as in (II). Now, by (2.67) and (2.68), and given the definition of x_k^*, for every $k \in \mathcal{K}_0 \cup \mathcal{K}_1$, the length of the interval $(x_k^* + \varepsilon, x_{k+1}^* - \varepsilon)$ is $x_{k+1}^* - x_k^* - 2\varepsilon > \zeta_1 - 2\varepsilon > \varepsilon$. Consequently, if $\bar{\mathbf{x}}^r \in X_K$ is the jump-point vector representation of $\bar{b}_r^\beta(\cdot)$, then $\bar{x}_{k+1}^r - \bar{x}_k^r > \varepsilon$ for all $k \in \mathcal{K}_0 \cup \mathcal{K}_1$, so that $\bar{\mathbf{x}}^r \in X_K^\varepsilon$.

Hence, $\bar{\mathbf{x}}^r \in X_K^\varepsilon$ is the unique solution to (note that the maximum is over X_K)

$$\max_{\mathbf{z} \in \mathbf{X}_K} \int_0^1 u_r^\beta(b_{\mathbf{z}}(x), x|\mathbf{x}^r, \mathbf{y}^r) f(x)\, dx. \tag{2.69}$$

A similar argument establishes that some $\bar{\mathbf{y}}^r \in X_K^\varepsilon$ is the unique solution to

$$\max_{\mathbf{z} \in \mathbf{X}_K} \int_0^1 u_r^\sigma(b_{\mathbf{z}}(x), x|\mathbf{x}^r, \mathbf{y}^r) f(x)\, dx. \tag{2.70}$$

[21]That is, if $\bar{p}$ maximizes $u_r^\beta(p, \bar{x}|\mathbf{x}^r, \mathbf{y}^r)$ on $\bar{\mathcal{P}}$, then for all $\underline{p} < \bar{p}$, if $\phi(x) \equiv u_r^\beta(\bar{p}, x|\mathbf{x}^r, \mathbf{y}^r) - u_r^\beta(\underline{p}, x|\mathbf{x}^r, \mathbf{y}^r)$ is zero at $x = \bar{x}$, then it is positive for $x > \bar{x}$ and negative for $x < \bar{x}$.

Then, for r large enough, $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r) \in C^0_{\varepsilon^2}$ and so, being feasible for (2.62), $(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$ must be the unique solution to (2.62). If this is the case, then for all r large enough, $\Psi^r_\varepsilon(\mathbf{x}^r, \mathbf{y}^r) = \{(\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)\}$ and so we must have $(\mathbf{x}^r, \mathbf{y}^r) = (\bar{\mathbf{x}}^r, \bar{\mathbf{y}}^r)$. Hence, $b_{\mathbf{x}^r}(\cdot) = b_{\bar{\mathbf{x}}^r}(\cdot) = \bar{b}^\beta_r(\cdot)$ maximizes a buyer's ex ante payoff among all measurable bidding functions, and similarly for $b_{\mathbf{y}^r}(\cdot)$ and a seller. Hence, $(b_{\mathbf{x}^r}(\cdot), b_{\mathbf{y}^r}(\cdot))$ is a double-auction equilibrium for $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$. Moreover, because $(\mathbf{x}^r, \mathbf{y}^r) \in C_{\varepsilon^2}$ and $\varepsilon < 1$, the equilibrium is nontrivial.

Because $\mathbf{x}^r \to \hat{\mathbf{x}}$, and because, for r large enough, the steps of $b_{\mathbf{x}^r}(\cdot)$ are bounded away from zero, each of $b_{\mathbf{x}^r}(\cdot)$ converges uniformly to $b_{\hat{\mathbf{x}}}(\cdot)$. By the choice of $\bar{\Delta}$ and because $b_{\hat{\mathbf{x}}}(\cdot)$ is an equilibrium of $\mathcal{E}(\alpha, v, f, g, \Delta)$, we have $\sup_{x\in[x(0),x(1))} |b_{\hat{\mathbf{x}}}(x) - v(x, \omega(x))| < \eta_0$. Moreover, because $b_{\bar{\mathbf{x}}^r}(x) = k\Delta$ for all x and k as in (II), and because $v(\hat{x}_{k_1}, 0) > (k_1 - 1)\Delta + \bar{\varepsilon}$ and $v(\hat{x}_{k_L}, 1) < k_L\Delta - \bar{\varepsilon}$ by Lemma 2.8, the construction of $\mathbf{x}^*$ ensures that $|b_{\bar{\mathbf{x}}^r}(x) - v(x, \omega(x))| < \Delta < \bar{\Delta} < \eta_0$ on $[0, \hat{x}_{k_1}]$ and on $[\hat{x}_{k_L}, 1]$. Hence, $|b_{\hat{\mathbf{x}}}(x) - v(x, \omega(x))| < \eta_0$ for all $x \in [0, 1]$. So, to complete the proof, we must establish (I) and (II). We begin with (I).

For all r large enough and for all $x \in [0, 1]$, no $p \in \bar{\mathcal{P}}$ such that $k_1\Delta < p < k_{L-1}\Delta$ and $p \notin \{k_1\Delta, \dots, k_{L-1}\Delta\}$ maximizes $u(p, x|\hat{\mathbf{x}})$ over $\bar{\mathcal{P}}$, by Lemma 2.13. Hence, the same holds true for $u(p, x|\hat{\mathbf{x}}, \hat{\mathbf{y}}) = u(p, x|\hat{\mathbf{x}})$, because $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are outcome-equivalent. However, then by Lemma 2.18, the same holds true for $u^\beta_r(p, x|\mathbf{x}^r, \mathbf{y}^r)$ for r large enough. Suppose that $\bar{p} \in \{k_1\Delta, \dots, k_{L-1}\Delta\}$ and that $\underline{p} < \bar{p}$. By Lemma 2.16, the length of each step of the price function induced by $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is less than $\bar{\eta}$. Consequently, because $u(p, x|\hat{\mathbf{x}}, \hat{\mathbf{y}}) = u(p, x|\hat{\mathbf{x}})$, Lemma 2.5 and the choice of $\bar{\eta}$ and Δ imply that there is a positive C^1 function $d(\cdot)$ such that

$$\frac{d}{dx}\frac{u(\bar{p}, x|\hat{\mathbf{x}}, \hat{\mathbf{y}}) - u(\underline{p}, x|\hat{\mathbf{x}}, \hat{\mathbf{y}})}{d(x)} > \bar{\eta}.$$

However, then, for all r large enough and all $x \in [0, 1]$,

$$\frac{d}{dx}\frac{u^\beta_r(\bar{p}, x|\mathbf{x}^r, \mathbf{y}^r) - u^\beta_r(\underline{p}, x|\mathbf{x}^r, \mathbf{y}^r)}{d(x)} > \frac{\bar{\eta}}{2},$$

by Lemma 2.18. Hence, the desired strict single-crossing property holds for all $p \in \bar{\mathcal{P}}$ between $k_1\Delta$ and $k_{L-1}\Delta$. To complete the proof of (I) it remains to consider $p < k_1\Delta$ and $p > k_{L-1}\Delta$. We will consider the former only, the latter being similar.[22]

[22]The analogous proof for $p > k_{L-1}\Delta$ employs (2.66) in a manner similar to that in which we will eventually employ (2.65) here, but (2.66) will not otherwise be explicitly employed in the present proof.

Suppose $\bar{p} \in \{k\Delta : k \in \mathcal{K}_0, k < k_1\}$ and $\underline{p} < \bar{p}$. For $p \in \bar{\mathcal{P}}$, let $\Pr(\tilde{B}^r_{m_r} = p$ and trade$|\mathbf{x}^r, \mathbf{y}^r, x)$ denote, from a buyer's perspective when he bids p, the probability that the m_rth highest bid of the others is equal to p and that he will trade, conditional on his signal being x, and given that the remaining $n_r - 1$ buyers employ the bid function $b_{\mathbf{x}^r}(\cdot)$ and all m_r sellers employ the bid function $b_{\mathbf{y}^r}(\cdot)$. Define

$$\phi_r(x) = \frac{u^\beta_r(\bar{p}, x|\mathbf{x}^r, \mathbf{y}^r) - u^\beta_r(\underline{p}, x|\mathbf{x}^r, \mathbf{y}^r)}{\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, x)}. \tag{2.71}$$

Define $\Pr(\tilde{B}^r_{m_r} = p$ and trade$|\mathbf{x}^r, \mathbf{y}^r, \omega)$ precisely as $\Pr(\tilde{B}^r_{m_r} = p$ and trade$|\mathbf{x}^r, \mathbf{y}^r, x)$ except that the former conditions on the state ω rather than on the signal x. Recall the choice of ε_1 and recall also that $\varepsilon < \varepsilon_1$. Hence, whenever $|x - y| \le \varepsilon^2$ and $\varepsilon(1-\varepsilon) \le x \le 1 - \varepsilon(1-\varepsilon)$, we have $F(x|\omega_0/2) > F(y|\omega_0)$. We first show that this implies

$$\frac{\Pr(\tilde{B}^r_{m_r} = \underline{p}|\mathbf{x}^r, \mathbf{y}^r, \omega)}{\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)} \to 0 \quad \text{uniformly in} \quad \omega \in [0,1] \tag{2.72}$$

and

$$\frac{\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \bar{\omega})}{\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \underline{\omega})} \to 0 \tag{2.73}$$

for every $\bar{p} \in \{k\Delta : k \in \mathcal{K}_0$ and $k < k_1\}$, every $\underline{p} < \bar{p}$, and every $\bar{\omega}, \underline{\omega} \in [0,1]$ such that $2\underline{\omega} < \omega_0 < \bar{\omega}$.

Let us begin with (2.72). Given that the buyer bids p and given that $B^r_{m_r} = p$, the probability that the buyer trades must be at least $1/(n_r + 1)$, because no more than $m_r - 1$ agents bid above p and no more than $n_r + 1$, including the buyer, bid p. Consequently, if $\bar{p} = \bar{k}\Delta$ and $\underline{p} = \underline{k}\Delta$, then

$$\begin{aligned}\frac{\Pr(\tilde{B}^r_{m_r} = \underline{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)}{\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)} &\le \frac{\Pr(\tilde{B}^r_{m_r} = \underline{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)}{\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)\frac{1}{n_r+1}} \\ &= (n_r + 1)\frac{\Pr(\tilde{B}^r_{m_r} = \underline{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)}{\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)}.\end{aligned}$$

Let $a_r = \min(x^r_{\underline{k}}, y^r_{\underline{k}})$, $b_r = \max(x^r_{\underline{k}+1}, y^r_{\underline{k}+1})$, $c_r = \max(x^r_{\bar{k}}, y^r_{\bar{k}})$, and $d_r = \min(x^r_{\bar{k}+1}, y^r_{\bar{k}+1})$. Clearly, $a_r \le b_r \le c_r$. Note that, because $\bar{k} \in \mathcal{K}_0$ and $(\mathbf{x}^r, \mathbf{y}^r) \in C_{\varepsilon^2} \cap (X^\varepsilon_K \times X^\varepsilon_K)$, $d_r - c_r = \min(x^r_{\bar{k}+1}, y^r_{\bar{k}+1}) - \max(x^r_{\bar{k}}, y^r_{\bar{k}}) \ge \varepsilon(1-\varepsilon) > 0$ for all r. Hence, $a_r \le b_r \le c_r < d_r < x(0)$. Indeed, d_r is bounded away from $x(0)$ because $x^r_{\bar{k}+1} \to \hat{x}_{\bar{k}+1}$ and $\bar{k} < k_1$ implies $\hat{x}_{\bar{k}+1} \le \hat{x}_{k_1} < x(0)$.

For the m_rth highest bid of the others to be equal to $\underline{k}\Delta$, the m_rth highest *signal* of the others must lie in $[a_r, b_r]$. Hence, $\Pr(\tilde{B}^r_{m_r} = \underline{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega) \le \Pr(\tilde{X}^r_{m_r} \in [a_r, b_r]|\mathbf{x}^r, \mathbf{y}^r, \omega)$, where $\tilde{X}^r_{m_r}$ is the m_rth highest signal of the other $n_r + m_r - 1$ agents in the rth economy. Also, if $X^r_{m_r} \in [c_r, d_r]$, then it is necessarily the case that $B^r_{m_r} = \bar{k}\Delta$. Hence, $\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega) \ge \Pr(\tilde{X}^r_{m_r} \in [c_r, d_r]|\mathbf{x}^r, \mathbf{y}^r, \omega)$. Putting these together yields

$$\begin{aligned}\frac{\Pr(\tilde{B}^r_{m_r} = \underline{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)}{\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)} &\le \frac{\Pr(\tilde{X}^r_{m_r} \in [a_r, b_r]|\mathbf{x}^r, \mathbf{y}^r, \omega)}{\Pr(\tilde{X}^r_{m_r} \in [c_r, d_r]|\mathbf{x}^r, \mathbf{y}^r, \omega)} \\ &= \frac{\int_{a_r}^{b_r} F^{n_r-1}(x|\omega) f(x|\omega)(1 - F(x|\omega))^{m_r-1}\,dx}{\int_{c_r}^{d_r} F^{n_r-1}(x|\omega) f(x|\omega)(1 - F(x|\omega))^{m_r-1}\,dx} \\ &= \frac{\int_{F(a_r|\omega)}^{F(b_r|\omega)} z^{n_r-1}(1-z)^{m_r-1}\,dz}{\int_{F(c_r|\omega)}^{F(d_r|\omega)} z^{n_r-1}(1-z)^{m_r-1}\,dz},\end{aligned}$$

where we have employed the change of variable $z = F(x|\omega)$.

Now, because $d_r - c_r \ge \varepsilon(1-\varepsilon)$ and $f(x|\omega) > 0$ for all x and ω, there exists $\beta > 0$ such that $\min_\omega(F(d_r|\omega) - F(c_r|\omega)) \ge \beta$ for all r. By Lemma 2.23, given this $\beta > 0$, there exists $\theta \in (0, 1)$ such that

$$\frac{\int_{F(a_r|\omega)}^{F(b_r|\omega)} z^{n_r-1}(1-z)^{m_r-1}\,dz}{\int_{F(c_r|\omega)}^{F(d_r|\omega)} z^{n_r-1}(1-z)^{m_r-1}\,dz} \le \frac{2}{\beta}\theta^{n_r+m_r-2} \tag{2.74}$$

because $d_r < x(0)$ implies $F(d_r|w) < F(x(0)|w) \le F(x(0)|0) = \alpha$, so that for all r large enough,

$$\frac{n_r - 1}{n_r + m_r - 2} \ge F(d_r|\omega).$$

Consequently,

$$(1 + n_r)\frac{\Pr(\tilde{B}^r_{m_r} = \underline{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)}{\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta|\mathbf{x}^r, \mathbf{y}^r, \omega)} \to 0$$

uniformly in $\omega \in [0, 1]$, proving (2.72).

Consider next (2.73). Let $a_r = \min(x^r_{\bar{k}}, y^r_{\bar{k}})$, $c_r = \max(x^r_{\bar{k}}, y^r_{\bar{k}})$, $d_r = \min(x^r_{\bar{k}+1}, y^r_{\bar{k}+1})$, and $b_r = \max(x^r_{\bar{k}+1}, y^r_{\bar{k}+1})$. Clearly, $a_r \le c_r \le d_r \le b_r$. Note that because $\bar{k} \in \mathcal{K}_0$ and $(\mathbf{x}^r, \mathbf{y}^r) \in C_{\varepsilon^2} \cap (X^\varepsilon_K \times X^\varepsilon_K)$ for all r, $d_r - c_r = \min(x^r_{\bar{k}+1}, y^r_{\bar{k}+1}) - \max(x^r_{\bar{k}}, y^r_{\bar{k}}) \ge \varepsilon(1-\varepsilon) > 0$ and $a_r \le c_r < d_r \le b_r < x(0)$. Indeed, b_r is bounded away from $x(0)$ because $x^r_{\bar{k}+1} \to \hat{x}_{\bar{k}+1}$, $y^r_{\bar{k}+1} \to \hat{y}_{\bar{k}+1}$, and $\bar{k} < k_1$ implies $\hat{x}_{\bar{k}+1} \le$

$\hat{x}_{k_1} < x(0)$ and $\hat{y}_{\bar{k}+1} \leq \hat{y}_{k_1} < x(0)$. Therefore, because ε was chosen so that $\varepsilon < 1 - x(0)$, we have $\varepsilon(1-\varepsilon) \leq d_r < x(0) < 1 - \varepsilon(1-\varepsilon)$.

Because $X^r_{m_r} \in [c_r, d_r]$ implies $B^r_{m_r} = \bar{k}\Delta$ implies $X^r_{m_r} \in [a_r, b_r]$ and, conditional on $B^r_{m_r}$ being equal to the buyer's bid of $\bar{k}\Delta$, the probability that the buyer trades is between 1 and $1/(n_r+1)$, we have

$$(2.75)\qquad \frac{\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \bar{\omega})}{\Pr(\tilde{B}^r_{m_r} = \bar{k}\Delta \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \underline{\omega})} \leq (n_r+1)\frac{\Pr(\tilde{X}^r_{m_r} \in [a_r, b_r]|\mathbf{x}^r, \mathbf{y}^r, \bar{\omega})}{\Pr(\tilde{X}^r_{m_r} \in [c_r, d_r]|\mathbf{x}^r, \mathbf{y}^r, \underline{\omega})}.$$

Now, as before,

$$(2.76)\qquad \frac{\Pr(\tilde{X}^r_{m_r} \in [a_r, b_r]|\mathbf{x}^r, \mathbf{y}^r, \bar{\omega})}{\Pr(\tilde{X}^r_{m_r} \in [c_r, d_r]|\mathbf{x}^r, \mathbf{y}^r, \underline{\omega})} = \frac{\int_{F(a_r|\bar{\omega})}^{F(b_r|\bar{\omega})} z^{n_r-1}(1-z)^{m_r-1}\,dz}{\int_{F(c_r|\underline{\omega})}^{F(d_r|\underline{\omega})} z^{n_r-1}(1-z)^{m_r-1}\,dz}.$$

Because $d_r - c_r > \varepsilon(1-\varepsilon)$, $F(d_r|\underline{\omega}) - F(c_r|\underline{\omega})$ is strictly positive and bounded away from zero. Also, $|b_r - d_r| = \max(x^r_{\bar{k}+1}, y^r_{\bar{k}+1}) - \min(x^r_{\bar{k}+1}, y^r_{\bar{k}+1}) \leq \varepsilon^2$ because $(\mathbf{x}^r, \mathbf{y}^r) \in C_{\varepsilon^2}$, and we have already noted that $\varepsilon(1-\varepsilon) \leq d_r < x(0) < 1 - \varepsilon(1-\varepsilon)$. Hence, given the choice of ω_0 and ε, $F(d_r|\omega_0/2) - F(b_r|\omega_0)$ is strictly positive and bounded away from zero. Therefore, because $2\underline{\omega} \leq \omega_0 \leq \bar{\omega}$, $F(d_r|\underline{\omega}) - F(b_r|\bar{\omega})$ is also strictly positive and bounded away from zero. Consequently, there exists $\beta > 0$ such that $F(d_r|\underline{\omega}) \geq \max(F(b_r|\bar{\omega}), F(c_r|\underline{\omega})) + \beta$ for all r.

Given this $\beta > 0$, there exists, by Lemma 2.23, $\theta \in (0,1)$ such that

$$\frac{\int_{F(a_r|\bar{\omega})}^{F(b_r|\bar{\omega})} z^{n_r-1}(1-z)^{m_r-1}\,dz}{\int_{F(c_r|\underline{\omega})}^{F(d_r|\underline{\omega})} z^{n_r-1}(1-z)^{m_r-1}\,dz} \leq \frac{2}{\beta}\theta^{n_r+m_r-2}$$

because $d_r < x(0)$ implies $F(d_r|\underline{w}) < F(x(0)|\underline{w}) \leq F(x(0)|0) = \alpha$ so that for all r large enough,

$$\frac{n_r - 1}{n_r + m_r - 2} \geq F(d_r|\underline{\omega}).$$

The remainder of the argument proceeds along the lines of the previous argument following (2.74). This proves (2.73).

If a buyer bids p, let $\Pr(\tilde{B}^r_{m_r} < p \text{ and trade}|\mathbf{x}, \mathbf{y}, \omega)$ denote the probability that the m_rth highest bid of the other $n_r + m_r - 1$ agents is strictly less than p and that the buyer trades at the market-clearing price (which might be p), conditional on the state ω and given that the other $n_r - 1$ buyers employ the bidding function $b_{\mathbf{x}}(\cdot)$ and all m_r sellers employ the bidding function $b_{\mathbf{y}}(\cdot)$. Then

$$\begin{aligned} &u_r^\beta(p, x|\mathbf{x}, \mathbf{y}) \\ &\quad= \int_0^1 (v(x, \omega) - p) \Pr(\tilde{B}^r_{m_r} = p \text{ and trade}|\mathbf{x}, \mathbf{y}, \omega) h(\omega|x)\, d\omega \\ &\qquad+ \int_0^1 (v(x, \omega) - \bar{\rho}^r(p, \mathbf{x}, \mathbf{y}, \omega)) \Pr(\tilde{B}^r_{m_r} < p \text{ and trade}|\mathbf{x}, \mathbf{y}, \omega) \\ &\qquad\quad \times h(\omega|x) d\omega, \end{aligned}$$

where $\bar{\rho}^r(p, \mathbf{x}, \mathbf{y}, \omega)$ is the expected value of the market-clearing price given the buyer's bid of p, the common strategy $b_{\mathbf{x}}(\cdot)$ of the remaining buyers, the common strategy $b_{\mathbf{y}}(\cdot)$ of the sellers, and conditional on (i) the m_rth highest bid of the other agents being strictly less than p, (ii) the buyer trading at the market-clearing price, and (iii) the state ω. The function $\bar{\rho}^r(p, \mathbf{x}, \mathbf{y}, \omega)$ is continuous in ω for each r and takes values in $[0, p]$ because $p > B^r_{m_r}$ implies that the market-clearing price is no higher than p.

Hence,

$$\begin{aligned} &u_r^\beta(\bar{p}, x|\mathbf{x}^r, \mathbf{y}^r) - u_r^\beta(\underline{p}, x|\mathbf{x}^r, \mathbf{y}^r) \\ &\quad= \int_0^1 (v(x, \omega) - \bar{p}) \Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega) h(\omega|x)\, d\omega \\ &\qquad+ \text{other terms}, \end{aligned}$$

where

$$\begin{aligned} &\text{other terms} \\ &\quad= \int_0^1 (v(x, \omega) - \bar{\rho}^r(\bar{p}, \mathbf{x}^r, \mathbf{y}^r, \omega)) h(\omega|x) \\ &\qquad\quad \times \Pr(\tilde{B}^r_{m_r} < \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)\, d\omega \\ &\qquad- \int_0^1 (v(x, \omega) - \underline{p}) h(\omega|x) \Pr(\tilde{B}^r_{m_r} = \underline{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)\, d\omega \\ &\qquad- \int_0^1 (v(x, \omega) - \bar{\rho}^r(\underline{p}, \mathbf{x}^r, \mathbf{y}^r, \omega)) h(\omega|x) \\ &\qquad\quad \times \Pr(\tilde{B}^r_{m_r} < \underline{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)\, d\omega. \end{aligned}$$

Hence, the absolute value of the "other terms" as well as the absolute value of its derivative with respect to x is bounded above by the sum of finitely many terms each of the form

$$\int_0^1 \phi(x,\omega)\Pr(\tilde{B}^r_{m_r} = p|\mathbf{x}^r, \mathbf{y}^r, \omega)\,d\omega,$$

where the $\phi(x,\omega)$, which may be distinct in each such term, is continuous in both variables and $p < \bar{p}$. Furthermore, the finite number of such terms is independent of r.

Therefore, $\phi_r(x)$ in (2.71) can be written as $\gamma_r(x) - \bar{p} + \xi_r(x)$, where

$$\gamma_r(x) = \frac{\int_0^1 (v(x,\omega))\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)h(\omega|x)\,d\omega}{\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, x)} \tag{2.77}$$

and

$$\xi_r(x) = \frac{\int_0^1 \phi(x,\omega)\Pr(\tilde{B}^r_{m_r} = p|\mathbf{x}^r, \mathbf{y}^r, \omega)\,d\omega}{\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, x)}. \tag{2.78}$$

Note that

$$\Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, x)$$
$$= \int_0^1 \Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)h(\omega|x)\,d\omega,$$

so that (2.77) can be rewritten as

$$\gamma_r(x) = \int_0^1 v(x,\omega)\frac{h(\omega|x)\varphi^r(\omega)}{\int_0^1 \varphi^r(\omega)h(\omega|x)\,d\omega}\,d\omega, \tag{2.79}$$

where $\varphi^r(\omega) = \Pr(\tilde{B}^r_{m_r} = \bar{p} \text{ and trade}|\mathbf{x}^r, \mathbf{y}^r, \omega)$. Consequently,

$$\gamma_r(x) = \int_0^1 v(x,\omega)\frac{h(\omega|x)}{\int_0^1 h(\omega|x)\,d\mu_r(\omega)}\,d\mu_r(\omega),$$

where the probability measure μ_r on $[0,1]$ is defined to have density $\varphi^r(\omega)/\int_0^1 \varphi^r(\omega)\,d\omega$ at ω. We next show that $\mu_r([0,\omega_0]) \to_r 1$.

It suffices to show that

$$\frac{\int_{\omega_0}^1 \varphi^r(\omega)\,d\omega}{\int_0^{\omega_0/2} \varphi^r(\omega)\,d\omega} \to_r 0.$$

The definitions of $\varphi^r(\omega)$ and (2.73) imply that $\varphi^r(\bar{\omega})/\varphi^r(\underline{\omega}) \to 0$ for all $\bar{\omega} > w_0$ and all $\underline{\omega} < \omega_0/2$. Hence,

$$\frac{\int_{\omega_0}^1 \varphi^r(\omega)\,d\omega}{\int_0^{\omega_0/2} \varphi^r(\omega)\,d\omega} = \int_{\omega_0}^1 \left[\int_0^{\omega_0/2} \frac{\varphi^r(\underline{\omega})}{\varphi^r(\bar{\omega})}\,d\underline{\omega}\right]^{-1} d\bar{\omega} \\ \to 0,$$

because the integral in square brackets, a function of $\bar{\omega}$, converges pointwise to $+\infty$.

Consequently, $\mu_r([0, \omega_0]) \to_r 1$ and the choice of ω_0 implies that $\gamma_r(x)$ satisfies the inequality in (2.65) for r large enough and all $x \in [0, 1]$. Following similar lines but using (2.72) rather than (2.73), it can be shown that $\xi_r(x)$ and $\xi_r'(x)$ converge uniformly to zero on [0, 1]. Hence, because γ_r and ξ_r and their derivatives are continuous, we have that for all r large enough, $|\phi_r(x) - (v(x, 0) - \bar{p})| < \zeta_0 \cdot \varepsilon_0$ and $|\phi_r'(x) - v_x(x, 0)| < \zeta_0 \cdot \varepsilon_0$ for all $x \in [0, 1]$. The latter inequality implies that $\phi_r'(x) > (1 - \varepsilon_0)\zeta_0 > 0$, because $\varepsilon_0 < 1$ and $v_x > \zeta_0$. This completes the proof of (I) as long as no $p < \underline{k}\Delta$ is a best reply for any signal and for r large enough. However, this follows by combining the two inequalities because when $\bar{p} = \underline{k}\Delta$, the former implies $\phi_r(0) > v(0, 0) - \underline{k}\Delta - \varepsilon_0\zeta_0 > 0$ (by the choice of ε_0), so that no $p < \underline{k}\Delta$ is as good as $\underline{k}\Delta$ when one's signal is zero, but then the latter implies that the same is true for all signals.

To see that condition (II) holds, note that it must hold for all $\bar{p} \in \{k\Delta : k \in \mathcal{K}_0, k < k_1\}$, because $|\phi_r(x) - (v(x, 0) - \bar{p})| < \zeta_0 \cdot \varepsilon_0$ and $v_x > \zeta_0$, and it must hold for $\bar{p} = k_1\Delta$ by a continuity argument, because $k_1\Delta$ is the unique best reply for $x \in (\hat{x}_{k_1}, \hat{x}_{k_1+1})$ at the limit $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. This proves (II) for $\mathcal{K}_0$. The proof for $\mathcal{K}_1$ is similar. *Q.E.D.*

Our final result proves (iii) and (iv) of RP Theorem 6.1. Indeed, Theorem 2.26 (i) and (ii) are slightly stronger than (iii) and (iv) of RP Theorem 6.1.

THEOREM 2.26: *Suppose that for every r and every $\Delta \in D$ a residual subset of* $(0, \bar{\Delta})$, $\mathcal{E}(n_r, m_r, v, f, g, \Delta)$ *possesses a double-auction equilibrium*, $(b_r^\Delta(\cdot), s_r^\Delta(\cdot))$ *such that* $\lim_r b_r^\Delta(x) = \lim_r s_r^\Delta(x) = b^\Delta(x)$ *uniformly in* $x \in [0, 1]$ *and where* $b^\Delta(\cdot)$ *is a double-auction equilibrium for the continuum economy* $\mathcal{E}(\alpha, v, f, g, \Delta)$.

For each equilibrium $(b_r^\Delta(\cdot), s_r^\Delta(\cdot))$, *let* $\tilde{P}_r^\Delta$ *denote the random double-auction market-clearing price and let* $\tilde{\beta}_r^\Delta$ *denote the random fraction of agents whose signals are strictly below the* m_r*th highest signal and who* (*inefficiently*) *end up with the good. Then*, *for every* $\varepsilon > 0$, *there exists* $\Delta' > 0$ *such that for every* $\Delta \in (0, \Delta') \cap D$, *the following equalities hold*:

(i) $\lim_r \Pr(|\tilde{P}_r^\Delta - v(x(\tilde{\omega}), \tilde{\omega})| < \varepsilon) = 1$;

(ii) $\lim_r \Pr(\tilde{\beta}_r^\Delta < \varepsilon) = 1$.

PROOF: For each equilibrium $(b_r^\Delta(\cdot), s_r^\Delta(\cdot))$ and conditional on each state $\omega \in [0, 1]$, let $\tilde{P}_r^\Delta(\omega)$ denote the random double-auction market-clearing price and let $\tilde{\beta}_r^\Delta(\omega)$ denote the random fraction of agents whose signals are strictly below the m_rth highest signal and who (inefficiently) end up with the good. Also, let $\tilde{Z}_r^\Delta$ denote the m_rth highest signal among all $n_r + m_r$ agents.

Suppose that for every $\varepsilon > 0$, there exists $\Delta' > 0$ such that for every $\Delta \in (0, \Delta') \cap D$ and almost every state of the good $\omega \in [0, 1]$, the following equalities hold:

(i′) $\lim_r \Pr(|\tilde{P}_r^\Delta(\omega) - v(x(\omega), \omega)| < \varepsilon | \omega) = 1$;

(ii′) $\lim_r \Pr(\tilde{\beta}_r^\Delta(\omega) < \varepsilon | \omega) = 1$.

Then, by (i′) and the dominated convergence theorem,

$$\begin{aligned}
&\Pr\big(|\tilde{P}_r^\Delta - v(x(\tilde{\omega}), \tilde{\omega})| < \varepsilon\big) \\
&\quad = \int_0^1 \Pr\big(|\tilde{P}_r^\Delta(\omega) - v(x(\omega), \omega)| < \varepsilon | \omega\big) g(\omega)\, d\omega \\
&\quad \to 1, \quad \text{as} \quad r \to \infty.
\end{aligned}$$

Similarly, (ii′) yields

$$\Pr(\tilde{\beta}_r^\Delta < \varepsilon) \to 1, \quad \text{as} \quad r \to \infty.$$

Hence, it suffices to establish (1′) and (2′).

Fix $\varepsilon > 0$. By Lemma 2.7, $P^\Delta(\omega) = b^\Delta(x(\omega)) \to v(x(\omega), \omega)$ uniformly in $\omega \in [0, 1)$ as $\Delta \to 0$ in D. Hence, we may choose $\Delta' \in (0, \bar{\Delta})$ such that

$$|P^\Delta(\omega) - v(x(\omega), \omega)| < \varepsilon$$

for every $\omega \in [0, 1)$ and every $\Delta \in (0, \Delta') \cap D$. Hence, for every $\omega \in [0, 1)$,

$$\begin{aligned}
&\Pr\big(|\tilde{P}_r^\Delta(\omega) - v(x(\omega), \omega)| < \varepsilon | \omega\big) \\
&\quad \geq \Pr(\tilde{P}_r^\Delta(\omega) = P^\Delta(\omega) | \omega) \\
&\quad \geq \Pr\big(\tilde{B}_{m_r-1}^{r,\Delta} = \tilde{B}_{m_r}^{r,\Delta} = \tilde{B}_{m_r+1}^{r,\Delta} = P^\Delta(\omega) | \omega\big).
\end{aligned}$$

By Lemma 2.19 the last expression converges to unity as $r \to \infty$ for all but finitely many $\omega \in [0, 1]$ (all probabilities are implicitly conditional on the equilibrium strategies $(b_r^\Delta(\cdot), s_r^\Delta(\cdot))$). Hence, (i′) holds.

It suffices to separately find Δ' so that (ii′) holds. For each $\Delta \in D$, the limit equilibrium $b^\Delta(\cdot)$ for $\mathcal{E}(\alpha, v, f, g, \Delta)$ is right-continuous and continuous at $x = 1$ by definition (and by construction; see the proof of Theorem 2.25). Let $\bar{\mathcal{P}} = \{0, \Delta, 2\Delta, \ldots, K\Delta\}$, where $(K - 1)\Delta < v(1, 1) \leq K\Delta$ and the dependence of K on Δ is suppressed. Then $b^\Delta(\cdot)$ is uniquely determined by its nondecreasing vector of jump points $\mathbf{x}^\Delta = (x_1^\Delta, \ldots, x_K^\Delta) \in X_K$.

Fix $\varepsilon > 0$. By Lemma 2.6, we may choose $\Delta' > 0$ such that for all $\Delta \in (0, \Delta') \cap D$, the vector of jump points that determine $b^{\Delta}(\cdot)$ satisfies

$$\max_{\omega\in[0,1]}\big(F(x_{k+1}^{\Delta}|\omega) - F(x_k^{\Delta}|\omega)\big) < \varepsilon/2 \tag{2.80}$$

whenever $(x_k^{\Delta}, x_{k+1}^{\Delta})$ and $(x(0), x(1))$ have nonempty intersection.

Fix $\Delta \in (0, \Delta') \cap D$. Because $\alpha \in (0, 1)$ and because $F(x|\omega)$ is strictly increasing in x and strictly decreasing in ω for every state $\omega \in [0, 1]$, except perhaps finitely many, there exists precisely one $k = 0, 1, \ldots, K$ such that

$$F(x_k^{\Delta}|\omega) < \alpha < F(x_{k+1}^{\Delta}|\omega), \tag{2.81}$$

where $x_0^{\Delta} = 0$ and $x_{K+1}^{\Delta} = 1$. Choose any state $\omega^0 \in [0, 1]$ other than one of the finitely many exceptional ones and suppose that ω^0 and k^0 satisfy (2.81). It suffices to show that $\lim_r \Pr(\tilde{\beta}_r^{\Delta}(\omega^0) < \varepsilon|\omega^0) = 1$.

Because $F(x(\omega)|\omega) = \alpha$ for all ω, (2.81) implies that $x(\omega^0) \in (x_{k^0}^{\Delta}, x_{k^0+1}^{\Delta})$, so that $(x_{k^0}^{\Delta}, x_{k^0+1}^{\Delta})$ and $(x(0), x(1))$ have nonempty intersection. Hence, by (2.80),

$$F(x_{k^0+1}^{\Delta}|\omega^0) - F(x_{k^0}^{\Delta}|\omega^0) < \varepsilon/2. \tag{2.82}$$

Suppose that the state is ω^0 and consider for each r the equilibrium $(b_r^{\Delta}(\cdot), s_r^{\Delta}(\cdot))$. Let $\tilde{\gamma}_r^{\Delta}$ be the random fraction of agents whose signals are in $(x_{k^0}^{\Delta}, x_{k^0+1}^{\Delta})$ and let $\tilde{\eta}_r^{\Delta}$ be the random fraction of agents who receive the good and whose signals are less than $x_{k^0}^{\Delta}$. Note that if the agents' signals are such that the realization of $\tilde{Z}_r^{\Delta}$, the m_rth highest signal, is less than $x_{k^0+1}^{\Delta}$, then the realizations of $\tilde{\gamma}_r^{\Delta}$, $\tilde{\eta}_r^{\Delta}$, and $\tilde{\beta}_r^{\Delta}(\omega^0)$ must satisfy

$$\gamma_r^{\Delta} + \eta_r^{\Delta} \geq \beta_r^{\Delta}(\omega^0). \tag{2.83}$$

Now, techniques similar to those in the proofs of Lemmas 2.55 and 2.56 can be employed to establish three equalities:

(a) $\lim_r \Pr(\tilde{Z}_r^{\Delta} < x_{k^0+1}^{\Delta}|\omega^0) = 1$;

(b) $\lim_r \Pr(\tilde{\gamma}_r^{\Delta} < \varepsilon/2|\omega^0) = 1$;

(c) $\lim_r \Pr(\tilde{\eta}_r^{\Delta} < \varepsilon/2|\omega^0) = 1$.

Informally, (a) follows because $x(\omega^0)$, the αth percentile of $F(\cdot|\omega^0)$, is strictly less than $x_{k^0+1}^{\Delta}$ by (2.81), and the m_rth highest signal converges with probability 1 to $x(\omega^0)$ by the law of large numbers as r tends to infinity because $n_r/(n_r + m_r) \to \alpha$. The limit in (b) also follows by the law of large numbers, using (2.82). Finally, (c) too follows from the law of large numbers, because buyers and sellers with signals less than $x_{k^0}^{\Delta}$ bid less than $k\Delta$ in the limit, and, by (2.81), $k\Delta$ is the market-clearing price with probability 1 in the limit.

Hence,

$$\begin{aligned}\lim_r \Pr(\tilde{\beta}_r^{\Delta}(\omega^0) < \varepsilon|\omega^0) &= \lim_r \Pr\bigl(\tilde{\beta}_r^{\Delta}(\omega^0) < \varepsilon \text{ and } \tilde{Z}_r^{\Delta} < x_{k^0+1}^{\Delta}|\omega^0\bigr)\\ &\geq \lim_r \Pr(\tilde{\gamma}_r^{\Delta} + \tilde{\eta}_r^{\Delta} < \varepsilon \text{ and } \tilde{Z}_r^{\Delta} < x_{k^0+1}^{\Delta}|\omega^0)\\ &= \lim_r \Pr(\tilde{\gamma}_r^{\Delta} + \tilde{\eta}_r^{\Delta} < \varepsilon|\omega^0)\\ &\geq \lim_r \Pr(\tilde{\gamma}_r^{\Delta} < \varepsilon/2 \text{ and } \tilde{\eta}_r^{\Delta} < \varepsilon/2|\omega^0)\\ &= 1,\end{aligned}$$

where the first and third lines follow from (a), the second line follows from (2.83), and the fourth line follows from (b) and (c). *Q.E.D.*

3. NONEXISTENCE OF MONOTONE BEST REPLIES IN DOUBLE AUCTIONS: AN EXAMPLE

We present here a finite-agent example in which one agent's only best reply to the nondecreasing strategies of the others is decreasing. Agents' bids are restricted to the discrete grid of prices $\mathcal{P} = \{0, 1, 2, \ldots\}$.

In Section 5 of RP, two potential sources for the failure of monotone best replies are discussed: the strategic effect and the rationing effect. An example in which nondecreasing best replies fail due to the rationing affect was also provided. Now, because the rationing effect arises only when ties occur with positive probability, it is absent when agents employ strictly increasing bidding functions, the distribution of private information is atomless, and the bid space is a continuum (the "standard" model). Consequently, the rationing effect alone is unlikely to be a serious deterrent to establishing the existence of nondecreasing equilibria in standard finite double auctions. The more serious deterrent comes from the strategic effect.

Our purpose here is to provide a discrete bid-space example in which nondecreasing best replies fail and only the strategic effect is present. Because the strategic effect is present even in the standard model and even when strictly increasing bidding functions are employed, our example suggests that it is not possible to establish the existence of nondecreasing best replies in finite double auctions with interdependent values and affiliated private information, even if values are private (as they are in our subsequent example).

To isolate the strategic effect, we assume here that the agent we focus on—a buyer—is never rationed. That is, this buyer's demand is filled whenever the price is less than or equal to his bid.

Consider a market with seven sellers, one buyer, and another "undecided" buyer who is contemplating his best response to the rest of the market. Bids must be nonnegative integers. Because all agents employ step functions, we may assume, without loss, that the signals are discrete. Let $X =$

$\{0, 1, 2, 3, 4, 5, 6\}$ denote the set of possible signals. Also, we suppose that there are two states of the good, $\omega = 0$ and 1, each being equally likely.

All agents have private values with value function $v(x) = 5.16 + x/600$. Let the conditional distribution of signals be given by

$$\Pr(x|\omega = 0) = \begin{cases} \varepsilon, & \text{if } x = 0, \\ (1-\varepsilon-\varepsilon^2)0.262001, & \text{if } x = 1, \\ (1-\varepsilon-\varepsilon^2)0.002625, & \text{if } x = 2, \\ (1-\varepsilon-\varepsilon^2)0.242958, & \text{if } x = 3, \\ (1-\varepsilon-\varepsilon^2)0.468673, & \text{if } x = 4, \\ (1-\varepsilon-\varepsilon^2)0.023743, & \text{if } x = 5, \\ \varepsilon^2, & \text{if } x = 6 \end{cases}$$

and

$$\Pr(x|\omega = 1) = \begin{cases} \varepsilon^2, & \text{if } x = 0 \\ (1-\varepsilon-\varepsilon^2)0.188757, & \text{if } x = 1, \\ (1-\varepsilon-\varepsilon^2)0.001892, & \text{if } x = 2, \\ (1-\varepsilon-\varepsilon^2)0.176519, & \text{if } x = 3, \\ (1-\varepsilon-\varepsilon^2)0.353832, & \text{if } x = 4, \\ (1-\varepsilon-\varepsilon^2)0.279000, & \text{if } x = 5, \\ \varepsilon, & \text{if } x = 6. \end{cases}$$

It is easy to check that the affiliation property is satisfied.

Let x_1 denote the undecided buyer's signal. Note that, in the limit as $\varepsilon \to 0$, the signal $x_1 = 0$ identifies the state as $\omega = 0$ and the signal $x_1 = 6$ identifies the state as $\omega = 1$. Hence, in the limit as $\varepsilon \to 0$, when the undecided buyer receives a signal of $x_1 = 0$, he infers that the others' signals are independent and identically distributed according to

$$\Pr(x|x_1 = 0) = \begin{cases} 0.262001, & \text{if } x = 1, \\ 0.002625, & \text{if } x = 2, \\ 0.242958, & \text{if } x = 3, \\ 0.468673, & \text{if } x = 4, \\ 0.023743, & \text{if } x = 5, \end{cases}$$

and when he receives a signal of $x_1 = 6$, he infers that the others' signals are independent and identically distributed according to

$$\Pr(x|x_1 = 6) = \begin{cases} 0.188757, & \text{if } x = 1, \\ 0.001892, & \text{if } x = 2, \\ 0.176519, & \text{if } x = 3, \\ 0.353832, & \text{if } x = 4, \\ 0.279000, & \text{if } x = 5. \end{cases}$$

Assume all agents in the market adopt the monotone strategies

$$s(x) = x + 1 \quad \text{and} \quad b(x) = x - 1$$

and that the market-clearing price is determined by the seventh highest bid.[23] If the seventh and eighth highest bids are equal and equal to the undecided buyer's bid, then the undecided buyer receives a unit at that price with probability 1. This eliminates the rationing effect and so isolates the strategic effect.

We are interested in comparing the undecided buyer's best responses, first when his signal is $x_1 = 0$ and then when it goes up to $x_1 = 6$, in the limit as $\varepsilon \to 0$. Let $u(0, p)$ and $u(6, p)$ denote the undecided buyer's expected payoff when his signal $x_1 = 0$ ($x_1 = 6$) and he submits a bid of p. Recall that $v(0) = 5.16$, $v(6) = 5.17$, and that, given the undecided buyer's two signals, in the limit as $\varepsilon \to 0$, the others' signals are independent and identically distributed according to $\Pr(x|x_1 = 0)$ and $\Pr(x|x_1 = 6)$. Because the undecided buyer's incentives will be strict under these limiting conditional distributions, they will also be strict for $\varepsilon > 0$ sufficiently small.

Note that given the strategies of the others, the set of possible market-clearing prices is $P = \{2, 3, 4, 5, 6\}$. Direct computations show that[24]

$$u(0, 2) = 2.3221,$$
$$u(0, 3) = 2.3228,$$
$$u(0, 4) = 2.1441,$$
$$u(0, 5) = 2.1146,$$
$$u(0, 6) = 2.1146$$

and

$$u(6, 2) = 1.6814,$$
$$u(6, 3) = 1.675,$$
$$u(6, 4) = 1.6716,$$
$$u(6, 5) = 1.5134,$$
$$u(6, 6) = 1.5109.$$

[23]Formally, negative bids (i.e., $b(0) = 0 - 1 = -1$) are not allowed, but because the undecided buyer regards the signal $x = 0$ to have probability 0, this is irrelevant here. Alternatively, one can suppose that the other buyer employs the strategy $b(x) = \max(0, x - 1)$. The result would be the same.

[24]We wish to thank Oren Rigbi for carrying out the programming required to obtain the parameters and payoffs in this example.

Thus, the "undecided" buyer's unique best response when his signal is 0 is $p = 3$, while it is $p = 2$ when his signal is 6. (Note that because the undecided buyer receives the good whenever the market-clearing price is weakly below his bid, his payoff from bidding $p \geq 7$ is identical to his payoff from bidding $p = 6$. Also, his payoff from bidding $p = 0$ or 1 is zero because he is sure not to trade.)

4. APPROXIMATING THE DENSITY IN RP SECTION 5.2

In Section 5.2 of RP, the following conditional density function is employed for convenience: $f(x|\omega) = 3/2$ if $(x, \omega) \in ([0, 2/3] \times [0, 1/2)) \cup ([1/3, 1] \times [1/2, 1])$ and $f(x|\omega) = 0$ otherwise. We show here that this conditional density function can be uniformly approximated by one that satisfies RP Assumptions A.1 and A.2.

For $\varepsilon > 0$ and $a \in [0, 1 - \varepsilon)$, let $\lambda_\varepsilon^a : [0, 1] \to [0, 1]$ be any strictly positive C^1 function with strictly positive derivative such that $\lambda_\varepsilon^a(x) \leq \varepsilon$ for $x \leq a$ and $\lambda_\varepsilon^a(x) \geq 1 - \varepsilon$ for $x \geq a + \varepsilon$. For $(x, \omega) \in [0, 1]^2$, define

$$h_\varepsilon(x, \omega) = [1 - \lambda_\varepsilon^{1/2}(\omega)]\lambda_\varepsilon^{1/3}(1 - x) + \lambda_\varepsilon^{1/2}(\omega)\lambda_\varepsilon^{1/3}(x)$$

and define

$$f_\varepsilon(x|\omega) = \frac{h_\varepsilon(x, \omega)}{\int_0^1 h_\varepsilon(x, \omega)\, dx}.$$

Then $f_\varepsilon(x|\omega)$ satisfies RP Assumptions A.1 and A.2 for every $\varepsilon < 1/2$, and $f_\varepsilon(x|\omega)$ converges uniformly to $f(x|\omega)$ as $\varepsilon \to 0$.

5. A MEASURE-MOTIVATED CONCEPT OF GENERICITY

We now show that the main result in RP, namely Theorem 6.1, also holds when the topological notion of genericity—residual sets—is replaced by a measure-motivated notion. To make such a statement precise, we require a notion of "full measure" for the infinite-dimensional Banach space C^1 of continuously differentiable value functions $v(x, \omega)$ on $[0, 1]^2$ endowed with the supremum metric $\|v\| = \max_{x,\omega}[v(x, \omega) + v_x(x, \omega) + v_\omega(x, \omega)]$. Unfortunately, there is no analogue of Lebesgue measure in such spaces. However, it *is* possible to limit oneself to defining a class of sets that generalizes the idea of Lebesgue measure zero in Euclidean spaces. One such class of sets is defined in Christensen (1974).

Let X be a separable Banach space.[25] Christensen (1974) defines a measurable subset A of X to be a *Haar zero* set if there exists a probability measure

[25] A Banach space, namely a complete normed linear space, is separable if it has a countable dense subset. The space of continuous functions on [0, 1] with the supremum norm is an example, as is any Euclidean space.

μ on X such that $\mu(A+x)=0$ for all $x \in X$.[26,27] Following Hunt, Sauer, and Yorke (1996), we shall call the complement of a Haar zero set *prevalent*. Furthermore, when Y is a subset of X with nonempty interior and $Z = X' \cap Y$ for some prevalent subset X' of X, we say that *Z is a prevalent subset of Y*. We then have the following:

THEOREM 5.1: *Theorem* 6.1 *of RP remains valid when "residual" is everywhere replaced by "prevalent" and where the topology on C^1 is induced by the metric* $\|v\| = \max_{x,\omega}[v(x,\omega) + v_x(x,\omega) + v_\omega(x,\omega)]$.

REMARK 1: Because a prevalent subset of $(0, \bar{\Delta})$ is a subset of full Lebesgue measure (i.e., of measure $\bar{\Delta}$), the second occurrence of "residual" in RP Theorem 6.1 can equivalently be replaced by "full Lebesgue measure."

To prove Theorem 5.1, it suffices to prove that Lemma 2.13 remains valid when "residual" is replaced with "prevalent." This is because the conclusions of Theorems 2.25 and 2.26 will then also hold when "residual" is replaced with "prevalent." Thus, it suffices to establish the following:

LEMMA 5.2: *The conclusion of Lemma* 2.13 *remains valid when "residual" is replaced with "prevalent*."

PROOF: Recall that $V \subseteq C^1$ is the set of value functions $v(x,\omega)$ that satisfy RP Assumptions A.3 and A.4. Let us say that $v \in V$ is *regular* at $L \geq 2$ if the statement

$$(*) \qquad \begin{array}{l} (2.42) \text{ and } (2.43) \text{ hold for every } \Delta \in D, \text{ for every} \\ k_0 < k_1 < \cdots < k_L, \text{ and for every } 0 = x_0 < x_1 < \cdots < x_{L+1} = 1 \\ \text{such that } x_2 > x(0),\ x_{L-1} < x(1) \text{ and such that (2.41) holds} \end{array}$$

is satisfied for some full Lebesgue measure subset D of $\mathbb{R}$. Let us say that $v \in V$ is *regular at $L \geq 2$ and $\Delta \in \mathbb{R}$* if $(*)$ is satisfied when $D = \{\Delta\}$.

For each integer $L \geq 2$, let $V_L = \{v \in V : v \text{ is not regular at } L\}$.[28] Let $C^1_* = \{v \in C^1 : v(0,0) \neq 0\}$. Because V has nonempty interior in C^1, it suffices to show that $V^0 = (V \cap C^1_*) \setminus \bigcup_{L=2}^{\infty} V_L$ is a prevalent subset of V. Because C^1_* is a

[26]In Euclidean spaces, it can be shown that the Haar zero sets coincide with the sets of Lebesgue measure zero.

[27]Recently, Hunt, Sauer, and Yorke (1992) rediscovered Christensen's definition. Their "shy" set is equivalent to Christensen's Haar zero set. Andersen and Zame (2000) take these equivalent concepts a step further by generalizing them to cover subtle, yet economically relevant, situations in which the ambient space is a "small" convex subset of a linear topological space. In the present more straightforward setting in which the ambient space is a Banach space, the definitions of Anderson and Zame, Hunt, Sauer, and Yorke, and Christensen, coincide. We thank George Mailath for calling our attention to Andersen and Zame's (2000) notion of "shyness."

[28]It is straightforward to show that V_L is a Borel subset of C^1.

prevalent subset[29] of C^1 and because countable unions of Haar zero sets are Haar zero sets (Christensen (1974)), it suffices to show that each V_L is a Haar zero subset of C^1.

Fix $L \geq 2$. Lemma 2.12 implies that for every $v \in V$, the set $\{(\varepsilon, \Delta) \in \mathbb{R}^{L+1}: v + \varepsilon \mathbf{p}_L$ is not regular at L and $\Delta\}$ has Lebesgue measure zero. By Fubini's theorem, for every $v \in V$ and almost every $\varepsilon \in \mathbb{R}^L$, $\{\Delta \in \mathbb{R}: v + \varepsilon \mathbf{p}_L$ is not regular at L and $\Delta\}$ has Lebesgue measure zero. Consequently, for all $v \in V$, $\{\varepsilon \in \mathbb{R}^L: v + \varepsilon \mathbf{p}_L \in V_L\}$ has Lebesgue measure zero.

Define the probability measure μ on C^1 as follows. For any Borel subset A of C^1, let $\mu(A) = \gamma\{\varepsilon \in \mathbb{R}^L: \varepsilon \mathbf{p}_L \in A\}$, where γ is the standard normal Gaussian measure in $\mathbb{R}^L$. Fix any $z \in C^1$. Then

$$\mu(V_L + z) = \gamma\{\varepsilon \in \mathbb{R}^L: \varepsilon \mathbf{p}_L \in V_L + z\} = \gamma\{\varepsilon \in \mathbb{R}^L: -z + \varepsilon \mathbf{p}_L \in V_L\}.$$

Now, if for every $\varepsilon \in \mathbb{R}^L$, $-z + \varepsilon \mathbf{p}_L \notin V_L$, then clearly $\mu(V_L + z) = 0$. Otherwise, there exists $\varepsilon^0 \in \mathbb{R}^L$ such that $-z + \varepsilon^0 \mathbf{p}_L \in V_L$. Let $E = \{\varepsilon \in \mathbb{R}^L: (-z + \varepsilon^0 \mathbf{p}_L) + \varepsilon \mathbf{p}_L \in V_L\}$ and note that $\{\varepsilon \in \mathbb{R}^L: -z + \varepsilon \mathbf{p}_L \in V_L\} = E + \varepsilon^0$, so that $\mu(V_L + z) = \gamma(E + \varepsilon^0)$. By the conclusion of the previous paragraph, E has Lebesgue measure zero. The translation invariance of Lebesgue measure implies that $E + \varepsilon^0$ also has Lebesgue measure zero. Consequently, because Gaussian and Lebesgue measure are mutually absolutely continuous, $\mu(V_L + z) = 0$. Because $z \in C^1$ was arbitrary, we conclude that V_L is a Haar zero subset of C^1. *Q.E.D.*

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[29]For any Borel subset A of C^1, define $\mu(A) = \gamma(\varepsilon \in \mathbb{R}: \varepsilon \mathbf{1} \in A\}$, where γ is the standard normal Gaussian measure in $\mathbb{R}$ and where $\mathbf{1}$ denotes the C^1 function $v(x, \omega) = 1$ all x and ω. Letting C_0^1 denote the complement of C_*^1, it is immediate that $\mu(C_0^1 + z) = 0$ for every $z \in C^1$ because $\varepsilon \mathbf{1} \in C_0^1 + z$ for at most one value of ε, namely $\varepsilon = z(0, 0)$.