Econometrica Supplementary Material

SUPPLEMENT TO "ROBUST INFERENCE ON INFINITE AND GROWING DIMENSIONAL TIME-SERIES REGRESSION"

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THIS SUPPLEMENT contains the remaining proofs of theorems and lemmas.

APPENDIX S.A: AN EXPONENTIAL INEQUALITY FOR PARTIAL SUMS OF WEAKLY DEPENDENT RANDOM MATRICES

We develop a stochastic order for a matrix partial sum. Closely related results can be found in Theorems 4.1 and 4.2 of Chen and Christensen (2015), who establish such bounds for full matrix sums as opposed to partial sums. Our first theorem is a Fuk–Nagaev-type inequality, using a coupling approach similar to Dedecker and Prieur (2004), Chen and Christensen (2015), and Rio (2017).

THEOREM ST.A.1: *Let $\{\xi_i\}_{i\in\mathbb{Z}}$ be a β-mixing sequence with support $\mathfrak{X}$ and rth mixing coefficient $\beta(r)$ and let $\Xi_{i,n}=\Xi_n(\xi_i)$, for each i, where $\Xi_n:\mathfrak{X}\to\mathbb{R}^{d_1\times d_2}$ is a sequence of measurable $d_1\times d_2$ matrix-valued functions. Assume $E(\Xi_{i,n})=0$ and $\|\Xi_{i,n}\|\le R_n$, for each i, set*

$$s_n^2=\max_{1\le i,j\le n}\max\{\|E(\Xi_{i,n}\Xi_{j,n}')\|,\|E(\Xi_{i,n}'\Xi_{j,n})\|\},$$

and define $S_k=\sum_{l=1}^k\Xi_{l,n}$. Then, for any integer q such that $1<q\le n/2$ and $\varrho\ge qR_n$,

$$P\Big(\sup_{1\le k\le n}\|S_k\|>4\varrho\Big)\le\left(\left[\frac{n}{q}\right]+1\right)\beta(q)+2(d_1+d_2)\exp\left(\frac{-\varrho^2/2}{nqs_n^2+qR_n\varrho/3}\right).$$

The required stochastic order now follows by a choice of ϱ in Theorem ST.A.1.

COROLLARY SC.A.1: *Under the conditions of Theorem ST.A.1, if q is chosen as a function of n such that $(n/q)\beta(q)=o(1)$ and $R_n\sqrt{q\log(d_1+d_2)}=o(s_n\sqrt{n})$ then*

$$\sup_{1\le k\le n}\|S_k\|=O_p\big(s_n\sqrt{nq\log(d_1+d_2)}\big)$$

PROOF OF THEOREM ST.A.1: For $i=1,\dots,[n/q]$, define $U_i=\sum_{j=iq-q+1}^{iq}\Xi_{j,n}$ and $U_{[n/q]+1}=\sum_{j=[n/q]q}^{n}\Xi_{j,n}$. Now, for an integer j that differs from an integer multiple of

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q by at most $[q/2]$, we have $\sup_{1\le k\le n}\|S_k\| \le 2[q/2]R_n + \sup_{j>0}\|\sum_{i=1}^j U_i\|$. If q is even (resp., odd) then $q = 2k$ (resp., $q = 2k+1$) for some positive integer k, implying $[q/2] = [2k/2] = k$ (resp., $[q/2] = [(2k+1)/2] = k$) whence $2[q/2]R_n \le qR_n$ (resp., $2[q/2]R_n \le (q-1)R_n$). Thus, because $\varrho \ge qR_n$,

$$P\Big(\sup_{1\le k\le n}\|S_k\| > 4\varrho\Big) \le P\big(2[q/2]R_n > \varrho\big) + P\left(\sup_{j>0}\left\|\sum_{i=1}^j U_i\right\| \ge 3\varrho\right)$$
$$= P\left(\sup_{j>0}\left\|\sum_{i=1}^j U_i\right\| \ge 3\varrho\right), \tag{SA.1}$$

so it suffices to prove that

$$P\left(\sup_{j>0}\left\|\sum_{i=1}^j U_i\right\| \ge 3\varrho\right) \le \left(\left[\frac{n}{q}\right]+1\right)\beta(q) + 2(d_1+d_2)\exp\left(\frac{-\varrho^2/2}{nqs_n^2 + qR_n\varrho/3}\right).$$

Enlarging the probability space as needed, by Lemma 5.1 (Berbee's lemma) of Rio (2017) there is a sequence ξ_i^*, $1\le i\le [n/q]+1$, such that:

(a) The random variable x_i^* is distributed as x_i for each $1\le i\le [n/q]+1$.
(b) The sequences ξ_{2i}^*, $1\le 2i\le [n/q]+1$, and ξ_{2i-1}^*, $1\le 2i-1\le [n/q]+1$, comprised of independent random variables.
(c) $P(\xi_i \ne \xi_i^*) \le \beta(q+p)$ for $1\le i\le [n/q]+1$.

Denote $\Xi_{i,n}^* = \Xi_n(\xi_i^*)$, and define U_i^* in the obvious manner. Then we have

$$\sup_{j>0}\left\|\sum_{i=1}^j U_i\right\| \le \sum_{i=1}^{[n/q]+1}\|U_i - U_i^*\| + \sup_{j>0}\left\|\sum_{i=1}^j U_{2i}^*\right\| + \sup_{j>0}\left\|\sum_{i=1}^j U_{2i-1}^*\right\|. \tag{SA.2}$$

Now, by (c), we have

$$P\left(\sum_{i=1}^{[n/q]+1}\|U_i - U_i^*\| \ge \varrho\right) = P\left(\sum_{i=1}^{[n/q]+1}\|U_i - U_i^*\| \ge \left[\sum_{i=1}^{[n/q]+1}\varrho/\big([n/q]+1\big)\right]\right)$$
$$\le \sum_{i=1}^{[n/q]+1} P\big(\|U_i - U_i^*\| \ge \varrho/\big([n/q]+1\big)\big)$$
$$\le \big([n/q]+1\big)\beta(q+p),$$

while for all $1\le i\le [n/q]+1$ the matrices $U_i^* = \sum_{j=iq-q+1}^{iq}\Xi_{j,n}^*$ satisfy $\|U_i^*\| \le qR_n$ and

$$\max_{1\le j\le n}\max\left\{\left\|E\left(\sum_{i=1}^j U_i U_i^{*'}\right)\right\|, \left\|E\left(\sum_{i=1}^j U_i^{*'} U_i^*\right)\right\|\right\} \le nqs_n^2.$$

Furthermore, the sequence $\mathcal{U}_j = \sum_{i=1}^j U_{2i}^*$ is a matrix martingale (because U_{2i}^* is an independent sequence and $E\mathcal{U}_j = 0$) with difference sequence $\mathcal{U}_j - \mathcal{U}_{j-1} = U_{2j}^*$. Thus, by

Corollary 1.3 of Tropp (2011),

$$P\left(\sup_{j>0}\left\|\sum_{i=1}^{j}U_{2i}^{*}\right\|\geq\varrho\right)\leq(d_1+d_2)\exp\left(\frac{-\varrho^2/2}{nqs_n^2+qR_n\varrho/3}\right). \tag{SA.3}$$

The third term on the RHS of (SA.2) is bounded similarly, whence the claim follows. *Q.E.D.*

PROOF OF COROLLARY SC.A.1: In Theorem ST.A.1, take $\varrho=Cs_n\sqrt{nq\log(d_1+d_2)}$ for a sufficiently large constant C. Then the claim follows by the condition $(n/q)\beta(q)=o(1)$ and because $R_n\sqrt{q\log(d_1+d_2)}=o(s_n\sqrt{n})$. To verify that ϱ satisfies that requirement of Theorem ST.A.1, note that the latter condition implies $Cs_n\sqrt{n}\geq R_n\sqrt{q\log(d_1+d_2)}$ for sufficiently large n, so $\varrho\geq qR_n\log(d_1+d_2)\geq qR_n$ for sufficiently large n, assuming $d_1+d_2\geq e\approx 2.72$. The latter condition fails only if the $\Xi_{i,n}$ are scalar. *Q.E.D.*

APPENDIX S.B: FOR SECTION 3

We first present an initial approximation of $\mathcal{Q}_n(\gamma)$.

THEOREM ST.B.1: *Let Assumptions* 1–3 *hold, and*

$$\lambda_n^{-4}\sqrt{p}\big(\lambda_n^{-1}\varkappa_p+\upsilon_p\big)+\lambda_n^{-6}p^{-1}\to 0\quad \text{as } n\to\infty, \tag{SB.1}$$

Then $\mathcal{Q}_n(\gamma)-(\mathcal{R}_n(\gamma)-p)/\sqrt{2p}=o_p(1)$.

PROOF: Much of the details are delegated to Lemmas SL.B.2–SL.B.10. In particular, we show in Lemma SL.B.6 that

$$\mathcal{Q}_n(\gamma)=\frac{n\varepsilon' A(\gamma)'B(\gamma)^{-1}A(\gamma)\varepsilon-p}{\sqrt{2p}}+o_p(1). \tag{SB.2}$$

Then note that

$$\big(X^*(\gamma)'M_X X^*(\gamma)\big)^{-1}=n^{-1}\big(I-\hat{M}^{-1}\hat{S}(\gamma)\big)^{-1}\hat{S}(\gamma)^{-1}, \tag{SB.3}$$

and

$$X^*(\gamma)'M_X\varepsilon=X^*(\gamma)'\varepsilon-\hat{S}(\gamma)\hat{M}^{-1}X'\varepsilon, \tag{SB.4}$$

because $n^{-1}X^*(\gamma)'X=\hat{S}(\gamma)$. Using (SB.3) and (SB.4), we may write $n\varepsilon' A(\gamma)'B(\gamma)^{-1}A(\gamma)\varepsilon/\sqrt{2p}$ as

$$\frac{n^{-1}R_1(\gamma)'R_2(\gamma)'B(\gamma)^{-1}R_2(\gamma)R_1(\gamma)}{\sqrt{2p}}, \tag{SB.5}$$

where $R_1(\gamma)=\hat{S}(\gamma)^{-1}X'^*(\gamma)\varepsilon-\gamma(1-\gamma)^{-1}\hat{M}^{-1}X'\varepsilon$ and $R_2(\gamma)=(I-\hat{M}^{-1}\hat{S}(\gamma))^{-1}$. By adding and subtracting terms, we can decompose (SB.5) as $\sum_{i=1}^{4}\Delta_i(\gamma)+\overline{\mathcal{R}_n}(\gamma)$, with

$$\Delta_1(\gamma)=\frac{\big(R_1(\gamma)-\overline{R}_1(\gamma)\big)'R_2(\gamma)'B(\gamma)^{-1}R_2(\gamma)R_1(\gamma)}{n\sqrt{2p}},$$

$$\Delta_2(\gamma) = \frac{\overline{R}_1(\gamma)' R_2(\gamma)' B(\gamma)^{-1} R_2(\gamma)\big(R_1(\gamma) - \overline{R}_1(\gamma)\big)}{n\sqrt{2p}},$$

$$\Delta_3(\gamma) = \frac{\overline{R}_1(\gamma)'\big(R_2(\gamma) - \gamma^{-1}I\big)' B(\gamma)^{-1} R_2(\gamma)\overline{R}_1(\gamma)}{n\sqrt{2p}},$$

$$\Delta_4(\gamma) = \frac{\overline{R}_1(\gamma)' B(\gamma)^{-1}\big(R_2(\gamma) - \gamma^{-1}I\big)\overline{R}_1(\gamma)}{\gamma n\sqrt{2p}},$$

where we write $\overline{R}_1(\gamma) = (1-\gamma)^{-1}\hat{M}^{-1}(\gamma\sum_{t=1}^{n}\varepsilon_t x_t - \sum_{t=1}^{[n\gamma]}\varepsilon_t x_t)$ and

$$\overline{\mathcal{R}_n}(\gamma) = \frac{\left(\sum_{t=1}^{[n\gamma]}\varepsilon_t x_t - \gamma\sum_{t=1}^{n}\varepsilon_t x_t\right)' \hat{M}^{-1} B(\gamma)^{-1}\hat{M}^{-1}\left(\sum_{t=1}^{[n\gamma]}\varepsilon_t x_t - \gamma\sum_{t=1}^{n}\varepsilon_t x_t\right)}{\gamma^2(1-\gamma)^2 n\sqrt{2p}}. \tag{SB.6}$$

By (SB.29), the term sandwiched between the parentheses in the numerator of (SB.6) is

$$\big(\hat{M}^{-1} - M^{-1}\big)B(\gamma)^{-1}\hat{M}^{-1} + M^{-1}B(\gamma)^{-1}\big(\hat{M}^{-1} - M^{-1}\big) + \gamma(1-\gamma)\Omega^{-1}. \tag{SB.7}$$

Substituting (SB.7) into (SB.6) yields three terms corresponding to the three terms in (SB.7). The first of these, multiplied by the outside terms in the sandwich formula in (SB.6), has modulus bounded by a constant times

$$\frac{n^{-1}\big(\|X'\varepsilon\|^2 + \|X^*(\gamma)'\varepsilon\|^2\big)\|\hat{M} - M\|\|B(\gamma)^{-1}\|\|M^{-1}\|\|\hat{M}^{-1}\|^2}{\sqrt{p}} = O_p\big(\lambda_n^{-4}\sqrt{p}\varkappa_p\big),$$

by Assumption 3 and Lemmas SL.B.2, SL.B.4, and also (SB.25), while the second is similarly shown to be negligible also. By (SB.1), we conclude that

$$\overline{\mathcal{R}_n}(\gamma) = \mathcal{R}_n(\gamma) + o_p(1),$$

indicating that the theorem is proved if $\Delta_i(\gamma) = o_p(1)$, $i = 1, 2, 3, 4$. But by previously used techniques and Lemmas SL.B.9 and SL.B.10, we readily conclude that

$$\big(\Delta_1(\gamma), \Delta_2(\gamma), \Delta_3(\gamma), \Delta_4(\gamma)\big) = O_p\big(\lambda_n^{-5}\sqrt{p}\varkappa_p\big),$$

which are all negligible by (SB.1), proving the theorem. *Q.E.D.*

Write $\tilde{\Omega}(\gamma) = n^{-1}\sum_{t=1}^{n} x_t(\gamma)x_t'(\gamma)\varepsilon_t^2$.

LEMMA SL.B.1: *Under Assumptions* 1–3, *and the conditions of Propositions B.*2 *or B.*1 *as applicable*,

$$\sup_{\gamma\in\Gamma}\big\|\hat{\Omega}(\gamma) - \tilde{\Omega}(\gamma)\big\| = O_p\left(\lambda_n^{-2}\min\left\{\frac{p^3}{n}, \frac{\vartheta_p^2 p}{n}\right\}\right), \tag{SB.8}$$

$$\sup_{\gamma\in\Gamma}\big\|\tilde{\Omega}(\gamma) - \bar{\Omega}(\gamma)\big\| = O_p\left(\frac{p}{\sqrt{n}}\right). \tag{SB.9}$$

PROOF OF LEMMA SL.B.1: The matrix inside the norm on the LHS of (SB.8) can be decomposed as $\sum_{i=1}^{5} U_i(\gamma)$, with

$$U_1(\gamma) = n^{-1}\sum_{t=1}^{n} x_t(\gamma)x_t'(\gamma)\big[x_t'(\gamma)\big(\delta - \hat{\delta}(\gamma)\big)\big]^2,$$

$$U_2(\gamma) = n^{-1}\sum_{t=1}^{n} x_t(\gamma)x_t'(\gamma)r_t^2,$$

$$U_3(\gamma) = 2n^{-1}\sum_{t=1}^{n} x_t(\gamma)x_t'(\gamma)\big[x_t'(\gamma)\big(\delta - \hat{\delta}(\gamma)\big)\big]\varepsilon_t,$$

$$U_4(\gamma) = 2n^{-1}\sum_{t=1}^{n} x_t(\gamma)x_t'(\gamma)\big[x_t'(\gamma)\big(\delta - \hat{\delta}(\gamma)\big)\big]r_t,$$

$$U_5(\gamma) = 2n^{-1}\sum_{t=1}^{n} x_t(\gamma)x_t'(\gamma)r_t\varepsilon_t.$$

Recall Lemma SL.B.3 for $\sup_{\gamma\in\Gamma}\|\delta - \hat{\delta}(\gamma)\| = O_p(\lambda_n^{-1}\sqrt{p/n})$. Now, since the maximum eigenvalue of a nonnegative definite symmetric matrix is less than equal to the trace,

$$\begin{aligned}\big\|U_1(\gamma)\big\| &\le n^{-1}\sum_{t=1}^{n}\big(x_t'(\gamma)x_t(\gamma)\big)^2\big(\delta - \hat{\delta}(\gamma)\big)'\big(\delta - \hat{\delta}(\gamma)\big)\\ &\le 2pn^{-1}\sum_{t=1}^{n}\sum_{j=1}^{p} x_{tj}^4\big\|\delta - \hat{\delta}(\gamma)\big\|^2 = O_p\big(\lambda_n^{-2}p^2\big)O_p(p/n),\end{aligned}$$

uniformly in γ, by the fact that $\sup_{t,j} Ex_{tj}^4 < \infty$ and (SB.27). In a similar fashion,

$$E\big\|U_2(\gamma)\big\| \le 2En^{-1}\sum_{t=1}^{n} x_t'x_t r_t^2 \le 2\big(E\big(x_t'x_t\big)^2 Er_t^4\big)^{1/2} = O\big(\lambda_n^{-2}p/\sqrt{n}\big).$$

Similarly, and using the fact that $E(|\varepsilon_t||x_t) \le \sqrt{E(\varepsilon_t^2|x_t)} = O(1)$, we obtain

$$\begin{aligned}\big\|U_3(\gamma)\big\| &\le 4n^{-1}\sum_{t=1}^{n}\big(x_t'x_t\big)^2|\varepsilon_t|\big\|\delta - \hat{\delta}(\gamma)\big\|^2 = O_p\big(\lambda_n^{-2}p^3/n\big),\\ \big\|U_4(\gamma)\big\| &\le 4n^{-1}\sum_{t=1}^{n}\big(x_t'x_t\big)^{3/2}\big\|\delta - \hat{\delta}(\gamma)\big\||r_t|\\ &\le 4\big\|\delta - \hat{\delta}(\gamma)\big\|\left(n^{-1}\sum_{t=1}^{n}\big(x_t'x_t\big)^2\right)^{3/4}\left(n^{-1}\sum_{t=1}^{n} r_t^4\right)^{1/4} = O_p\left(\sqrt{\frac{p}{n}}p^{3/2}\frac{\lambda_n^{-1}}{n^{1/4}}\right),\\ \big\|U_5(\gamma)\big\| &= 2n^{-1}\sum_{t=1}^{n}\big(x_t'x_t\big)|r_t\varepsilon_t| = O_p(p/\sqrt{n}),\end{aligned}$$

all uniformly in Γ. Thus, (SB.8) is established.

To show (SB.9), let x_{it}, $i=1,\dots,p$, be a typical element of x_t. Then any element of $\tilde{\Omega}(\gamma)-\bar{\Omega}(\gamma)$ is of the form $n^{-1}\sum_{t=1}^{n}x_{it}(\gamma)x_{jt}(\gamma)(\varepsilon_t^2-\sigma_t^2)$, $i,j=1,\dots,p$, and $\varepsilon_t^2-\sigma_t^2$ is an MDS by construction. Thus, it has mean zero and variance $n^{-2}\sum_{t=1}^{n}Ex_{it}^2(\gamma)x_{jt}^2(\gamma)E((\varepsilon_t^2-\sigma_t^2)^2|\mathcal{F}_{t-1})=O_p(n^{-1})$, by Assumption 1 and the boundedness of Ex_{it}^4. Thus, $E\|\tilde{\Omega}(\gamma)-\bar{\Omega}(\gamma)\|^2=O(p^2/n)$, and the claim in (SB.9) follows by Markov's inequality. *Q.E.D.*

We establish asymptotic normality of

$$\mathcal{S}_n(\gamma)=\frac{n^{-1}\sum_{s\neq t}g_t(\gamma)'\Omega^{-1}g_s(\gamma)\varepsilon_t\varepsilon_s}{\gamma(1-\gamma)\sqrt{2p}},\tag{SB.10}$$

recalling that $g_t(\gamma)=x_t1\{t/n\le\gamma\}-\gamma x_t$.

THEOREM ST.B.2: *Under Assumptions* 1–5 *and* (SB.1), $\mathcal{S}_n(\gamma)\xrightarrow{d}\sqrt{\mathcal{V}}\mathcal{Q}(\gamma)$, *as* $n\to\infty$, *pointwise in* γ.

PROOF OF THEOREM ST.B.2: First, note that $\mathcal{S}_n(\gamma)$ equals $[\gamma(1-\gamma)\sqrt{2p}]^{-1}$ times

$$\frac{1}{n}\sum_{\substack{s,t=1\\s\neq t}}^{[n\gamma]}x_t'\Omega^{-1}x_s\varepsilon_t\varepsilon_s-\frac{2\gamma}{n}\sum_{s=1}^{n}\sum_{\substack{t=1\\s\neq t}}^{[n\gamma]}x_t'\Omega^{-1}x_s\varepsilon_t\varepsilon_s+\frac{\gamma^2}{n}\sum_{\substack{s,t=1\\s\neq t}}^{n}x_t'\Omega^{-1}x_s\varepsilon_t\varepsilon_s,$$

and thus,

$$\begin{aligned}\mathcal{S}_n(\gamma)&=\frac{\sqrt{2}}{\gamma(1-\gamma)}\big[\mathcal{A}_n(\gamma)-\gamma\big[\mathcal{A}_n(1)+\mathcal{A}_n(\gamma)-\bar{\mathcal{A}}_n(\gamma)\big]+\gamma^2\mathcal{A}_n(1)\big],\\&=\sqrt{2}\bigg(\frac{\mathcal{A}_n(\gamma)}{\gamma}+\frac{\bar{\mathcal{A}}_n(\gamma)}{(1-\gamma)}-\mathcal{A}_n(1)\bigg),\end{aligned}$$

where

$$\mathcal{A}_n(\gamma)=\frac{1}{n\sqrt{p}}\sum_{s=2}^{[n\gamma]}\sum_{t=1}^{s-1}\xi_t'\xi_s,$$

$$\bar{\mathcal{A}}_n(\gamma)=\frac{1}{n\sqrt{p}}\sum_{s=[n\gamma]+1}^{n}\sum_{t=[n\gamma]+1}^{s-1}\xi_t'\xi_s,$$

and $\xi_t=\{\xi_{ti}\}_{i=1}^{p}=\Omega^{-1/2}x_t\varepsilon_t$ being an mds.

Next, we check the conditions of Corollary 3.1 in Hall and Heyde (1980) for $\mathcal{A}_n(\gamma)$, with similar steps holding for $\bar{\mathcal{A}}_n(\gamma)$ due to the symmetric nature of the processes. Writing $w_{ns}=\xi_s'\sum_{t=1}^{s-1}\xi_t/\sqrt{np}$ (a heterogeneous martingale difference array), we first check the second condition therein, namely

$$\sum_{s}E\big(w_{ns}^2|\mathcal{G}_{s-1}\big)/n-\mathcal{V}/2\xrightarrow{p}0.\tag{SB.11}$$

Let $\Delta_s = \Upsilon_s \Xi_s$. Then we want to show $n^{-2}p^{-1}\sum_s \operatorname{tr}\Delta_s - \mathcal{V}/2 \xrightarrow{p} 0$, but because $n^{-2}p^{-1}\sum_s E\operatorname{tr}\Delta_s \to \mathcal{V}/2$, it suffices to show

$$n^{-2}p^{-1}\sum_s(\operatorname{tr}\Delta_s - E\operatorname{tr}\Delta_s) \xrightarrow{p} 0. \tag{SB.12}$$

By Assumption 4, $P\{Z_n \le n^\nu\} \to 1$, where $Z_n = \max_{1\le t\le n}\overline{\lambda}(\Upsilon_t)$. Then, to prove (SB.12), let $d_n = I\{Z_n \le n^\nu\}$ and K_n denote the LHS in (SB.12). Write $K_n = K_n d_n + K_n(1-d_n)$ and note that $K_n(1-d_n) = o_p(1)$ since $P\{\|Z_n K_n(1-d_n)\| > 0\} \le P\{d_n \ne 1\} \to 0$.

Thus, it suffices to show (SB.12) for $d_n = 1$. The LHS of (SB.12) has variance

$$\begin{aligned} &n^{-4}p^{-2}\sum_s E(\operatorname{tr}\Delta_s - E\operatorname{tr}\Delta_s)^2 \\ &\quad + 2n^{-4}p^{-2}\sum_{s_1<s_2} E\big((\operatorname{tr}\Delta_{s_1} - E\operatorname{tr}\Delta_{s_1})(\operatorname{tr}\Delta_{s_2} - E\operatorname{tr}\Delta_{s_2})\big). \end{aligned} \tag{SB.13}$$

The first term in (SB.13) is bounded by $n^{-4}p^{-2}\sum_s E(\operatorname{tr}^2\Delta_s)$, and observe that

$$\begin{aligned} \sum_s E\big(\operatorname{tr}^2\Delta_s\big) &= \sum_s E\big(\operatorname{tr}^2(\Upsilon_s\Xi_s)\big) \le \sum_s E\big\{\overline{\lambda}^2(\Upsilon_s)\operatorname{tr}^2(\Xi_s)\big\} \\ &\le Cn^{2\nu}E\bigg(\sum_s \operatorname{tr}^2(\Xi_s)\bigg). \end{aligned} \tag{SB.14}$$

The above inequalities are obtained as follows: first, the matrix $\Xi_s = \sum_{t_1,t_2<s}\xi_{t_1}\xi'_{t_2}$ is symmetric and positive semidefinite as it equals $(\sum_{t<s}\Omega^{-1/2}x_t\varepsilon_t)(\sum_{t<s}\Omega^{-1/2}x_t\varepsilon_t)'$. Because Υ_s is also symmetric psd, Theorem 1 of Fang, Loparo, and Feng (1994) yields $\operatorname{tr}(\Upsilon_s\Xi_s) \le \overline{\lambda}(\Upsilon_s)\operatorname{tr}(\Xi_s)$, whence the remaining inequality follows by Assumption 4.

Because $\operatorname{tr}(\Omega^{-1/2}x_{t_1}x'_{t_2}\Omega^{-1/2}) = x'_{t_1}\Omega^{-1}x_{t_2}$, the right side of (SB.14) is

$$Cn^{2\nu}\sum_s \sum_{t_1,t_2<s;t_3,t_4<s} E\big(x'_{t_1}\Omega^{-1}x_{t_2}\varepsilon_{t_1}\varepsilon_{t_2}x'_{t_3}\Omega^{-1}x_{t_4}\varepsilon_{t_3}\varepsilon_{t_4}\big). \tag{SB.15}$$

The contribution to (SB.15) when $t_1 = t_2 = t_3 = t_4$ is

$$Cn^{2\nu}\sum_s\sum_{t<s}E\big(\big(x'_t\Omega^{-1}x_t\big)^2\varepsilon_t^4\big) \le Cn^{2\nu}\sum_s\sum_{t<s}\sum_{i,j=1}^p E\big(x_{it}^2x_{jt}^2\big) = O\big(n^{2\nu+2}p^2\big),$$

by Assumptions 1 and 3. Thus, this case contributes $O(n^{2\nu-2}) = o(1)$ to (SB.13). Next, the contribution to (SB.15) from the case $(t_1 = t_2) \ne (t_3 = t_4)$ is

$$\begin{aligned} &Cn^{2\nu}\sum_s\sum_{t_1<t_2<s}E\big(x'_{t_1}\Omega^{-1}x_{t_1}\varepsilon_{t_1}^2E\big(x'_{t_2}\Omega^{-1}x_{t_2}\varepsilon_{t_2}^2|\mathcal{G}_{t_2-1}\big)\big) \\ &\quad \le Cn^{2\nu}\sum_s\sum_{t_1<s}E\bigg(x'_{t_1}\Omega^{-1}x_{t_1}\varepsilon_{t_1}^2\sum_{t_2<s}\operatorname{tr}\Upsilon_{t_2}\bigg) \\ &\quad \le Cn^{2\nu+1}p\sum_{t_1\le n}E\bigg(x'_{t_1}\Omega^{-1}x_{t_1}\varepsilon_{t_1}^2\sum_{t_2\le n}\overline{\lambda}(\Upsilon_{t_2})\bigg) \end{aligned}$$

$$
\begin{aligned}
&\le Cn^{3\nu+2} p \sum_{t_1 \le n} \operatorname{tr}\bigl(E\bigl(x_{t_1} x_{t_1}' \varepsilon_{t_1}^2\bigr)\Omega^{-1}\bigr) \\
&= O\bigl(n^{3\nu+3} p^2\bigr),
\end{aligned}
$$

by Assumption 4, and because $\operatorname{tr} \Upsilon_{t_2} \le p\overline{\lambda}(\Upsilon_{t_2})$. Thus, this case contributes $O(n^{3\nu+3} p^2)$ to (SB.14) and, therefore, $O(n^{3\nu-1})$ to (SB.13).

The cases $(t_1 = t_3) \neq (t_2 = t_4)$ and $(t_1 = t_4) \neq (t_2 = t_3)$ similarly contribute a constant times

$$
\begin{aligned}
&n^{2\nu} \sum_s \sum_{t_1 \neq t_2} E\bigl(x_{t_1}' \Omega^{-1} x_{t_2} \varepsilon_{t_1} \varepsilon_{t_2}\bigr)^2 \\
&\quad \le n^{2\nu} \sum_s \sum_{t_1 \neq t_2} \bigl(E\bigl(x_{t_1}' \Omega^{-1} x_{t_1} \varepsilon_{t_1}^2\bigr)^2\bigr)^{1/2} \bigl(E\bigl(x_{t_2}' \Omega^{-1} x_{t_2} \varepsilon_{t_2}^2\bigr)^2\bigr)^{1/2} \\
&\quad = O\bigl(n^{2\nu+3} p^2\bigr),
\end{aligned}
\tag{SB.16}
$$

to (SB.15), using the Cauchy–Schwarz inequality. This ensures a negligible contribution of $O(n^{2\nu-1})$ to (SB.13). Finally,

$$
\begin{aligned}
&Cn^{2\nu} \sum_s \sum_{t_1, t_2 < s; t_3, t_4 < s}^{\neq} E\bigl(x_{t_1}' \Omega^{-1} x_{t_2} \varepsilon_{t_1} \varepsilon_{t_2} x_{t_3}' \Omega^{-1} x_{t_4} \varepsilon_{t_3} \varepsilon_{t_4}\bigr) \\
&\quad = O\left(n^{2\nu} \sum_s \sum_{t_1, t_2 < s; t_3, t_4 < s}^{\neq} \sum_{i,j=1}^{p} \bigl|E(x_{t_1,i}\varepsilon_{t_1} x_{t_2,i}\varepsilon_{t_2} x_{t_3,j}\varepsilon_{t_3} x_{t_4,j}\varepsilon_{t_4})\bigr|\right) \\
&\quad = O\left(n^{2\nu+1} p^2 \max_s \sum_{t_1, t_2 < s; t_3, t_4 < s}^{\neq} \max_{i,j=1,\dots,p} \bigl|E(x_{t_1,i}\varepsilon_{t_1} x_{t_2,i}\varepsilon_{t_2} x_{t_3,j}\varepsilon_{t_3} x_{t_4,j}\varepsilon_{t_4})\bigr|\right),
\end{aligned}
\tag{SB.17}
$$

where $\sum_{t_1,t_2<s;t_3,t_4<s}^{\neq}$ excludes all cases, which were considered before. Therefore, in view of (SB.14) and (SB.17), to establish negligibility of the first term in (SB.13) it suffices to show

$$
n^{2\nu-3} \max_s \max_{i,j=1,\dots,p} \sum_{t_1, t_2 < s; t_3, t_4 < s}^{\neq} \bigl|E(x_{t_1,i}\varepsilon_{t_1} x_{t_2,j}\varepsilon_{t_2} x_{t_3,k}\varepsilon_{t_3} x_{t_4,l}\varepsilon_{t_4})\bigr| = o(1).
\tag{SB.18}
$$

The summand on the LHS above is bounded by

$$
\begin{aligned}
&\bigl|E(x_{t_1,i}\varepsilon_{t_1} x_{t_2,i}\varepsilon_{t_2})\bigr| \bigl|E(x_{t_3,j}\varepsilon_{t_3} x_{t_4,j}\varepsilon_{t_4})\bigr| + \bigl|E(x_{t_1,i}\varepsilon_{t_1} x_{t_3,j}\varepsilon_{t_3})\bigr| \bigl|E(x_{t_2,i}\varepsilon_{t_2} x_{t_4,j}\varepsilon_{t_4})\bigr| \\
&\quad + \bigl|E(x_{t_1,i}\varepsilon_{t_1} x_{t_4,j}\varepsilon_{t_4})\bigr| \bigl|E(x_{t_2,i}\varepsilon_{t_2} x_{t_3,j}\varepsilon_{t_3})\bigr| \\
&\quad + \bigl|\operatorname{cum}_{iijj}(x_{t_1,i}\varepsilon_{t_1}, x_{t_2,i}\varepsilon_{t_2}, x_{t_3,j}\varepsilon_{t_3}, x_{t_4,j}\varepsilon_{t_4})\bigr| \\
&= \bigl|c_{ii}(t_1 - t_2)\bigr| \bigl|c_{jj}(t_3 - t_4)\bigr| + \bigl|c_{ij}(t_1 - t_3)\bigr| \bigl|c_{ij}(t_2 - t_4)\bigr| \\
&\quad + \bigl|c_{ij}(t_1 - t_4)\bigr| \bigl|c_{ji}(t_2 - t_3)\bigr| \\
&\quad + \bigl|\operatorname{cum}_{iijj}(x_{0,i}\varepsilon_0, x_{t_2-t_1,i}\varepsilon_{t_2-t_1}, x_{t_3-t_1,j}\varepsilon_{t_3-t_1}, x_{t_4-t_1,j}\varepsilon_{t_4-t_1})\bigr|.
\end{aligned}
\tag{SB.19}
$$

Because $\sum_{t_1,t_2} |c_{ij}(t_1-t_2)| \le n\sum_{t=-\infty}^{\infty} |c_{ij}(t)|$, by Assumption 5 and (SB.19) the LHS of (SB.18) is $O(n^{2(\nu+1)-3}) = O(n^{2\nu-1}) = o(1)$, as desired. Thus, the first term in (SB.13) is negligible, and by Assumption 4 we conclude the proof of (SB.12).

We now check the conditional Lindeberg condition

$$\text{For all } \eta > 0, \quad \sum_s E\big((w_{ns}/\sqrt{n})^2 1(|w_{ns}| > \eta)|\mathcal{G}_{s-1}\big) \xrightarrow{p} 0, \tag{SB.20}$$

in Hall and Heyde (1980), Corollary 3.1, for which we verify the sufficient Lyapunov condition

$$\sum_s E\big((w_{ns}/\sqrt{n})^4|\mathcal{G}_{s-1}\big) \xrightarrow{p} 0. \tag{SB.21}$$

The LHS of (SB.21) is positive and, by the law of iterated expectations, has mean

$$n^{-2}\sum_s E w_{ns}^4 \le n^{-1}\max_s E w_{ns}^4 = o(1),$$

the final bound coming due to a calculation similar to the proof of (SB.11). Specifically,

$$\begin{aligned} n^{-1}\max_s E|w_{ns}|^4 &\le \max_s E\bigg(E\big((\xi_s'\xi_s)^2|\mathcal{G}_{s-1}\big)\bigg(\sum_{t_1,t_2<s}\xi_{t_1}'\xi_{t_2}\bigg)^2\bigg)n^{-3}p^{-2} \\ &= O\big(n^{\omega+\nu-1}\big), \end{aligned} \tag{SB.22}$$

by Assumption 4 and because the steps involved in showing (SB.11) imply that $E(\sum_{t_1,t_2<s}\xi_{t_1}'\xi_{t_2})^2 = O(n^{\nu+2}p^2)$. Thus, because Assumption 4 also implies that $O(n^{\omega+\nu-1}) = o(1)$, (SB.20) is established. A similar proof holds for the asymptotic normality of $\bar{\mathcal{A}}_n(\gamma)$.

We finally derive the limiting covariance of $(\mathcal{A}_n(\gamma), \bar{\mathcal{A}}_n(\gamma))'$. Using Assumption 4, we first compute

$$\begin{aligned} E\big|\mathcal{A}_n(\gamma)\big|^2 &= \frac{1}{n}\sum_{s=1}^{[n\gamma]} E w_s^2 \\ &= \frac{1}{n^2}\sum_{s=1}^{[n\gamma]} s\left(\frac{1}{sp}\operatorname{tr}\sum_{t_1,t_2=1}^{s-1} E\big(\xi_s\xi_s'\xi_{t_1}\xi_{t_2}'\big)\right) \\ &= \frac{([n\gamma]+1)[n\gamma]}{2n^2}\lim_{s,p\to\infty}\left(\frac{1}{sp}\operatorname{tr}\sum_{t_1,t_2=1}^{s-1} E\big(\xi_s\xi_s'\xi_{t_1}\xi_{t_2}'\big)\right) + o(1) \\ &= \frac{\gamma^2\mathcal{V}}{2} + o(1), \end{aligned}$$

where $\mathcal{V}$ is given in (3.2). Next,

$$E\big|\bar{\mathcal{A}}_n(\gamma)\big|^2 = E\left|\frac{1}{n\sqrt{p}}\sum_{s=[n\gamma]+1}^{n}\sum_{t=[n\gamma]+1}^{s-1}\xi_t'\xi_s\right|^2$$

$$
\begin{aligned}
&= E\left|\frac{1}{n\sqrt{p}} \sum_{s=1}^{n-[n\gamma]} \sum_{t=1}^{s-1} \xi'_{t+[n\gamma]} \xi_{s+[n\gamma]}\right|^2 \\
&= \frac{(1-\gamma)^2 \mathcal{V}}{2} + o(1).
\end{aligned}
$$

Finally,

$$
E\big(\mathcal{A}_n(\gamma)\bar{\mathcal{A}}_n(\gamma)\big) = 0.
$$

Therefore, we conclude that

$$
\begin{pmatrix} \mathcal{A}_n(\gamma) \\ \bar{\mathcal{A}}_n(\gamma) \end{pmatrix} \xrightarrow{d} \sqrt{\frac{\mathcal{V}}{2}} \begin{pmatrix} W(\gamma) \\ \bar{W}(\gamma) \end{pmatrix},
$$

pointwise in $\gamma \in \Gamma$.

Finally, apply the continuous mapping theorem to get, pointwise in γ,

$$
\begin{aligned}
\mathcal{S}_n(\gamma) &= \sqrt{2}\left(\frac{\mathcal{A}_n(\gamma)}{\gamma} + \frac{\bar{\mathcal{A}}_n(\gamma)}{(1-\gamma)} - \mathcal{A}_n(1)\right) \\
&\xrightarrow{d} \sqrt{\mathcal{V}}\left(\frac{W(\gamma)}{\gamma} + \frac{\bar{W}(\gamma)}{(1-\gamma)} - W(1)\right) = \sqrt{\mathcal{V}}\mathcal{Q}(\gamma),
\end{aligned}
$$

where $\mathcal{Q}(\gamma)$ has a standard normal distribution for this given γ. *Q.E.D.*

We record some preliminary calculations useful for the sequel. Note that

$$
\hat{\delta}_2(\gamma) = A(\gamma)y = \delta_2 + A(\gamma)e = \delta_2 + A(\gamma)\varepsilon + A(\gamma)r.
$$

Because $\delta_2 = 0$ under $\mathcal{H}_0$, we have

$$
W_n(\gamma) = n(\varepsilon + r)' A'(\gamma) \hat{B}(\gamma)^{-1} A(\gamma)(\varepsilon + r), \tag{SB.23}
$$

where we recall that $\hat{B}(\gamma) = R\hat{M}(\gamma)^{-1}\hat{\Omega}(\gamma)\hat{M}(\gamma)^{-1}R'$.

LEMMA SL.B.2: *Under the conditions of Theorem ST.B.1, for all sufficiently large n,*

$$
\sup_{\gamma\in\Gamma} \|\hat{M}(\gamma)\| = O_p(1), \qquad \sup_{\gamma\in\Gamma} \|\hat{M}(\gamma)^{-1}\| = O_p\big(\lambda_n^{-1}\big).
$$

PROOF: Note that, by the triangle inequality,

$$
\|\hat{M}(\gamma)^{-1}\| \le \|\hat{M}(\gamma)^{-1}\| \|\hat{M}(\gamma) - M(\gamma)\| \|M(\gamma)^{-1}\| + \|M(\gamma)^{-1}\|,
$$

so

$$
\|\hat{M}(\gamma)^{-1}\|\big(1 - \|\hat{M}(\gamma) - M(\gamma)\| \|M(\gamma)^{-1}\|\big) \le \|M(\gamma)^{-1}\|,
$$

using the triangle inequality. Taking limits of the last displayed expression as $n \to \infty$ and using Assumption 3, the rate condition (SB.1) yields $\|\hat{M}(\gamma)^{-1}\| = O_p(\lambda_n^{-1})$. Next, noting that

$$\|\hat{M}(\gamma)\| \le \|\hat{M}(\gamma) - M(\gamma)\| + \|M(\gamma)\|,$$

the lemma follows by using Assumption 3. *Q.E.D.*

It is useful to first establish the stochastic order of $\|\delta - \hat{\delta}(\gamma)\|$.

LEMMA SL.B.3: *Under the conditions of Theorem* ST.B.1, $\sup_{\gamma\in\Gamma} \|\delta - \hat{\delta}(\gamma)\| = O_p(\lambda_n^{-1}\sqrt{p/n})$.

PROOF: Note that $\delta - \hat{\delta}(\gamma) = \hat{M}(\gamma)^{-1} n^{-1} \sum_{t=1}^n x_t(\gamma) e_t$ and that

$$\|\delta - \hat{\delta}(\gamma)\|^2 = O_p\left(\|\hat{M}(\gamma)^{-1}\|^2 n^{-2} \left\|\sum_{t=1}^n x_t(\gamma) e_t\right\|^2\right) = \lambda_n^{-2} O_p\left(n^{-2}\left\|\sum_{t=1}^n x_t(\gamma) e_t\right\|^2\right)$$
$$= \lambda_n^{-2} O_p\left(n^{-2}\left\|\sum_{t=1}^n x_t(\gamma)\varepsilon_t\right\|^2 + n^{-2}\|X(\gamma)' r\|^2\right),$$

uniformly in γ, by Lemma SL.B.2. Next, $E(n^{-2}\|\sum_{t=1}^n x_t(\gamma)\varepsilon_t\|^2)$ equals

$$E\left(n^{-2}\sum_{s,t=1}^n x_t'(\gamma) x_s(\gamma)\varepsilon_s\varepsilon_t\right), \tag{SB.24}$$

which is

$$n^{-2}\sum_{t=1}^n E\|x_t(\gamma)\|^2\sigma_t^2 + 2n^{-2}\sum_{s<t} E\big(x_t'(\gamma)x_s(\gamma)E\big(\varepsilon_s E(\varepsilon_t|\varepsilon_r, r<t)\big)\big)$$
$$= O_p(p/n), \tag{SB.25}$$

by Assumptions 1 and $Ex_t'(\gamma)x_t(\gamma) = O(p)$. Finally,

$$n^{-2}\|X(\gamma)' r\|^2 \le n^{-2}\|X(\gamma)\|^2\|r\|^2 = \overline{\lambda}\big(\hat{M}(\gamma)\big) n^{-1}\|r\|^2 = O_p(1/n), \tag{SB.26}$$

by (2.2) and Lemma SL.B.2. Therefore,

$$\sup_{\gamma\in\Gamma}\|\delta - \hat{\delta}(\gamma)\| = O_p\big(\lambda_n^{-1}\sqrt{p}/\sqrt{n}\big), \tag{SB.27}$$

by Markov's inequality. *Q.E.D.*

Observe that because

$$M(\gamma)^{-1} = \begin{bmatrix} (1-\gamma)^{-1}M^{-1} & (1-\gamma)^{-1}M^{-1} \\ (1-\gamma)^{-1}M^{-1} & [\gamma(1-\gamma)]^{-1}M^{-1} \end{bmatrix}, \tag{SB.28}$$

we have

$$B(\gamma)^{-1} = \gamma(1-\gamma)M\Omega^{-1}M. \tag{SB.29}$$

LEMMA SL.B.4: *Under the conditions of Theorem* ST.B.1,

$$\sup_{\gamma\in\Gamma}\{\underline{\lambda}(B(\gamma))\}^{-1} = O(\lambda_n^{-1}) \quad \textit{and} \quad \sup_{\gamma\in\Gamma}\overline{\lambda}(B(\gamma)) = O(\lambda_n^{-2}).$$

PROOF: $\{\underline{\lambda}(B(\gamma))\}^{-1} = \overline{\lambda}(B(\gamma)^{-1})$, which, using (SB.29), is bounded by

$$C\overline{\lambda}(M\Omega^{-1}M) = C\|M\Omega^{-1}M\| \leq C\overline{\lambda}(M)^2\underline{\lambda}(\Omega)^{-1} = O(\lambda_n^{-1}),$$

uniformly on the compact Γ, using Assumption 3(ii). For the second part of the claim, because (SB.29) implies $B(\gamma) = [\gamma(1-\gamma)]^{-1}M^{-1}\Omega M^{-1}$, it follows similarly that $\overline{\lambda}(B(\gamma))$ is uniformly bounded by a constant times[1]

$$\overline{\lambda}(M^{-1}\Omega M^{-1}) = \|M^{-1}\Omega M^{-1}\| \leq \underline{\lambda}(M)^{-2}\overline{\lambda}(\Omega) = O(\lambda_n^{-2}). \qquad \textit{Q.E.D.}$$

LEMMA SL.B.5: *Under the conditions of Theorem* ST.B.2,

$$\sup_{\gamma\in\Gamma}\|\hat{B}(\gamma)\| = O_p(\lambda_n^{-2}), \qquad \sup_{\gamma\in\Gamma}\|\hat{B}(\gamma)^{-1}\| = O_p(\lambda_n^{-1}).$$

PROOF: We show the second claim, the first following easily by the definition of $\hat{B}(\gamma)$. First, define $\tilde{B}(\gamma) = R\hat{M}(\gamma)^{-1}\Omega(\gamma)\hat{M}(\gamma)^{-1}R'$. We will use uniform bounds in the calculations without explicitly mentioning this in each step to simplify notation. Proceeding as in the proof of Lemma SL.B.2, we can write

$$\|\hat{B}(\gamma)^{-1}\|(1 - \|\hat{B}(\gamma) - \tilde{B}(\gamma)\|) \leq \|\tilde{B}(\gamma)^{-1}\|, \tag{SB.30}$$

$$\|\tilde{B}(\gamma)^{-1}\|(1 - \|\tilde{B}(\gamma) - B(\gamma)\|) \leq \|B(\gamma)^{-1}\|. \tag{SB.31}$$

Next, Lemma SL.B.2 implies

$$\begin{aligned}\|\hat{B}(\gamma) - \tilde{B}(\gamma)\| &\leq \|R\|^2\|\hat{M}(\gamma)^{-1}\|^2\|\hat{\Omega}(\gamma) - \Omega(\gamma)\| = O_p(\lambda_n^{-2}v_p) \\ &= o_p(1). \end{aligned} \tag{SB.32}$$

On the other hand, $\tilde{B}(\gamma) - B(\gamma)$ equals

$$R[\hat{M}(\gamma)^{-1}\Omega(\gamma)\hat{M}(\gamma)^{-1} - M(\gamma)^{-1}\Omega(\gamma)M(\gamma)^{-1}]R'.$$

By adding and subtracting terms inside the square brackets, this can be written as

$$\begin{aligned}&R[M(\gamma)^{-1}(\hat{M}(\gamma) - M(\gamma))\hat{M}(\gamma)^{-1}\Omega(\gamma)\hat{M}(\gamma)^{-1}]R' \\ &\quad + RM(\gamma)^{-1}\Omega(\gamma)M(\gamma)^{-1}(\hat{M}(\gamma) - M(\gamma))\hat{M}(\gamma)^{-1}R'. \end{aligned} \tag{SB.33}$$

[1]If $\underline{\lambda}(M\Omega^{-1}M) \geq \lambda_n$, the bound in this lemma becomes $O(\lambda_n^{-1})$.

By this fact, Assumption 3, Lemmas SL.B.1 and SL.B.2, and (SB.1), we deduce from (SB.33) that

$$\|\tilde{B}(\gamma) - B(\gamma)\| = O_p(\lambda_n^{-3}\varkappa_p) = o_p(1). \tag{SB.34}$$

The lemma now follows by taking limits of (SB.30) and (SB.31), and using (SB.32), (SB.34), and Lemma SL.B.4. *Q.E.D.*

LEMMA SL.B.6: *Under the conditions of Theorem* ST.B.2 *and* $\mathcal{H}_0$,

$$\frac{W_n(\gamma)}{\sqrt{2p}} = \frac{n\varepsilon' A(\gamma)' B(\gamma)^{-1} A(\gamma)\varepsilon}{\sqrt{2p}} + o_p(1).$$

PROOF: Recall the notation $\hat{M} = n^{-1}X'X$ and $\hat{S}(\gamma) = n^{-1}X'^{*}(\gamma)X(\gamma)$. Notice that from (SB.23) we obtain

$$\begin{aligned}\frac{W_n(\gamma)}{\sqrt{2p}} &= \frac{n\varepsilon' A(\gamma)'\hat{B}(\gamma)^{-1}A(\gamma)\varepsilon}{\sqrt{2p}} + \frac{2n\varepsilon' A(\gamma)'\hat{B}(\gamma)^{-1}A(\gamma)r}{\sqrt{2p}} \\ &\quad + \frac{nr' A(\gamma)'\hat{B}(\gamma)^{-1}A(\gamma)r}{\sqrt{2p}},\end{aligned} \tag{SB.35}$$

with r the $n \times 1$ vector with elements r_t. Begin with the modulus of the last term on the RHS of (SB.35). Recalling the relation in (SB.3) and (SB.4) for $A(\gamma)r$, we bound it by $Cn/\sqrt{2p}$ times

$$\begin{aligned}&O_p\big(\|n^{-1}X'r\|^2\|I - \hat{S}(\gamma)\hat{M}^{-1}\|^2\big\|\big(n^{-1}X^*(\gamma)'M_X X^*(\gamma)\big)^{-1}\big\|^2\|\hat{B}(\gamma)^{-1}\|\big). \\ &= O_p(\lambda_n^{-3}n^{-1}),\end{aligned} \tag{SB.36}$$

where Assumption 2 bounds the first term, Lemma SL.B.9 yields a bound for the second and third terms after expanding the third term by (SB.3), and the last term is $O_p(\lambda_n^{-1})$ Lemma SL.B.5. Thus, (SB.36) implies that the third term on the RHS of (SB.35) is $o_p(1)$.

We now show that the first term on the RHS of (SB.35) is

$$\frac{n\varepsilon' A(\gamma)' B(\gamma)^{-1} A(\gamma)\varepsilon}{\sqrt{2p}} + o_p(1). \tag{SB.37}$$

Indeed, as above,

$$\begin{aligned}\frac{n\varepsilon' A(\gamma)'(\hat{B}(\gamma)^{-1} - B(\gamma)^{-1})A(\gamma)\varepsilon}{\sqrt{2p}} &= \frac{n}{\sqrt{p}}O_p\big(\|n^{-1}X'\varepsilon\|^2\|\hat{B}(\gamma)^{-1} - B(\gamma)^{-1}\|\big) \\ &= \sqrt{p}O_p\big(\|B(\gamma)^{-1}\|\,\|B(\gamma) - \hat{B}(\gamma)\|\,\|\hat{B}(\gamma)^{-1}\|\big) \\ &= \lambda_n^{-4}\sqrt{p}O_p\big(\lambda_n^{-1}\|\hat{M}(\gamma) - M(\gamma)\| \qquad \text{(SB.38)} \\ &\quad + \|\hat{\Omega}(\gamma) - \Omega(\gamma)\|\big) \\ &= O_p\big(\lambda_n^{-4}\sqrt{p}(\lambda_n^{-1}\varkappa_p + \upsilon_p)\big), \qquad \text{(SB.39)}\end{aligned}$$

using equations (SB.32) and (SB.34). This is negligible by (SB.1).

For the second term on the RHS of (SB.35), apply the Cauchy–Schwarz inequality and the preceding two results. Then the second term becomes $o_p(1)$, establishing the lemma. *Q.E.D.*

Denote, for convenience, $C(\gamma) = [\gamma(1-\gamma)]^{-1}n^{-1}\Sigma^{\frac{1}{2}}G(\gamma)\Omega^{-1}G(\gamma)\Sigma^{\frac{1}{2}}$, where $\Sigma = \operatorname{diag}[\sigma_1^2, \dots, \sigma_n^2]$.

LEMMA SL.B.7: *Under the conditions of Theorem ST.B.2, any eigenvalue λ of $C(\gamma)$ satisfies*

$$P\big(\big|\lambda(\lambda-1)\big| < \eta\big) \to 1,$$

as $n \to \infty$, for any $\eta > 0$.

PROOF: We have

$$\begin{aligned} C(\gamma)^2 &= \big[\gamma(1-\gamma)\big]^{-1}n^{-1}\Sigma^{\frac{1}{2}}G(\gamma)\Omega^{-1}\big[\gamma(1-\gamma)\big]^{-1}n^{-1}G(\gamma)'\Sigma G(\gamma)\Omega^{-1}G(\gamma)'\Sigma^{\frac{1}{2}} \\ &= \big[\gamma(1-\gamma)\big]^{-1}n^{-1}\Sigma^{\frac{1}{2}}G(\gamma)\Omega^{-1}\Omega\Omega^{-1}G(\gamma)'\Sigma^{\frac{1}{2}} \\ &\quad + \big[\gamma(1-\gamma)\big]^{-1}n^{-1}\Sigma^{\frac{1}{2}}G(\gamma)\Omega^{-1}\big\{\big[\gamma(1-\gamma)\big]^{-1}n^{-1}G(\gamma)'\Sigma G(\gamma) - \Omega\big\} \\ &\quad \times \Omega^{-1}G(\gamma)'\Sigma^{\frac{1}{2}} \\ &= C(\gamma) + D(\gamma), \end{aligned}$$

say. We now prove that

$$\big\|D(\gamma)\big\| = o_p(1) \quad \text{as } n \to \infty. \tag{SB.40}$$

In view of Assumptions 1 and 3(i), to prove (SB.40) it suffices to show that

$$\big\|\big[\gamma(1-\gamma)\big]^{-1}n^{-1}G(\gamma)'\Sigma G(\gamma) - \Omega\big\| = o_p(1). \tag{SB.41}$$

But

$$\begin{aligned} n^{-1}G(\gamma)'\Sigma G(\gamma) &= n^{-1}(1-2\gamma)\sum_{t=1}^{[n\gamma]} x_t x_t'\sigma_t^2 + \gamma^2\Omega \\ &= (1-2\gamma)\left(n^{-1}\sum_{t=1}^{[n\gamma]} x_t x_t'\sigma_t^2 - \gamma\Omega\right) + \big[\gamma(1-\gamma)\big]\Omega, \end{aligned}$$

so (SB.41) follows if $\|n^{-1}\sum_{t=1}^{[n\gamma]} x_t x_t'\sigma_t^2 - \gamma\Omega\| = o_p(1)$, which is true by Assumption 3. Thus, (SB.40) is established.

Let λ be any eigenvalue of $C(\gamma)$ and w be the corresponding eigenvector, normalized to $\|w\| = 1$. Because $\lambda w = C(\gamma)w$, we have $\lambda C(\gamma)w = C(\gamma)^2 w = [C(\gamma) + D(\gamma)]w = \lambda w + D(\gamma)w$, implying $\lambda(\lambda-1)w = D(\gamma)w$. Thus,

$$\big|\lambda(\lambda-1)\big| = \big\|D(\gamma)w\big\| \le \big\|D(\gamma)\big\|. \tag{SB.42}$$

Then, for arbitrary $\eta > 0$,

$$P(|\lambda(\lambda-1)| < \eta) = P(\|D(\gamma)w\| < \eta) \geq P(\|D(\gamma)\| < \eta) \to 1, \quad \text{as } n \to \infty,$$

by (SB.40). This completes the proof. *Q.E.D.*

We have $\mathcal{R}_n(\gamma) = [\gamma(1-\gamma)]^{-1} n^{-1} \varepsilon' G(\gamma)' \Omega^{-1} G(\gamma)' \varepsilon$, which in turn equals

$$[\gamma(1-\gamma)]^{-1} n^{-1} \sum_{t,s=1}^{n} g_t(\gamma)' \Omega^{-1} g_s(\gamma) \varepsilon_t \varepsilon_s. \tag{SB.43}$$

Note that $\operatorname{tr}\{C(\gamma)\}$ is the sum of the eigenvalues of $C(\gamma)$, which is a symmetric matrix with rank p. Thus, in view of Lemma SL.B.7 it has p eigenvalues that approach 1 in probability, with the remainder approaching 0. Thus,

$$\frac{\mathcal{R}_n(\gamma) - \operatorname{tr}(C(\gamma))}{\sqrt{2p}} = \frac{\mathcal{R}_n(\gamma) - p}{\sqrt{2p}} + o_p(1), \tag{SB.44}$$

whence using (SB.43) we deduce that (SB.44) equals

$$\frac{n^{-1} \sum_{t=1}^{n} g_t(\gamma)' \Omega^{-1} g_t(\gamma) (\varepsilon_t^2 - \sigma_t^2) + n^{-1} \sum_{s \neq t} g_t(\gamma)' \Omega^{-1} g_s(\gamma) \varepsilon_t \varepsilon_s}{\gamma(1-\gamma)\sqrt{2p}}. \tag{SB.45}$$

LEMMA SL.B.8: *Under the conditions of Theorem ST.B.2,*

$$\sup_{\gamma \in \Gamma} n^{-1} \sum_{t=1}^{n} g_t(\gamma)' \Omega^{-1} g_t(\gamma) (\varepsilon_t^2 - \sigma_t^2) = o_p(1) \quad \text{as } n \to \infty. \tag{SB.46}$$

PROOF: Conditional on x_t, the LHS of (SB.46) has mean zero and variance

$$n^{-2} \sum_{t=1}^{n} (g_t(\gamma)' \Omega^{-1} g_t(\gamma))^2 E[(\varepsilon_t^2 - \sigma_t^2)^2] \tag{SB.47}$$

$$+ 2n^{-2} \sum_{s<t} g_s(\gamma)' \Omega^{-1} g_s(\gamma) g_t(\gamma)' \Omega^{-1} g_t(\gamma) E[(\varepsilon_t^2 - \sigma_t^2)(\varepsilon_s^2 - \sigma_s^2)]. \tag{SB.48}$$

The expectation in (SB.48) equals $E[(\varepsilon_t^2 - \sigma_t^2) E((\varepsilon_s^2 - \sigma_s^2) | \varepsilon_s)] = 0$, by Assumption 1. Also, by Assumption 1, (SB.47) is bounded by a constant times

$$n^{-2} \|\Omega^{-1}\|^2 \sum_{t=1}^{n} \|g_t(\gamma)\|^4 \leq n^{-2} \|\Omega^{-1}\|^2 \sum_{t=1}^{n} (\|x_t(\gamma)\|^4 + \gamma^4 \|x_t\|^4) = O_p\left(\lambda_n^{-2} \frac{p^2}{n}\right),$$

uniformly in γ, the last equality following by Assumption 3(i). *Q.E.D.*

LEMMA SL.B.9: *Under the conditions of Theorem ST.B.2, as $n \to \infty$,*

$$\|(I - \hat{M}^{-1} \hat{S}(\gamma))^{-1} - \gamma^{-1} I\| = O_p(\lambda_n^{-1} \varkappa_p).$$

PROOF: First, note that $\|(I-\hat{M}^{-1}\hat{S}(\gamma))-\gamma I\|$ equals

$$\begin{aligned}&\big\|(1-\gamma)I-\hat{M}^{-1}\big(\hat{S}(\gamma)-(1-\gamma)M\big)-(1-\gamma)\hat{M}^{-1}M\big\|\\&\quad\leq C\big\|\hat{M}^{-1}\big\|\big(\|\hat{M}-M\|+\big\|\hat{S}(\gamma)-(1-\gamma)M\big\|\big)\\&\quad=O_p\big(\lambda_n^{-1}\varkappa_p\big),\end{aligned}$$

by Assumptions 3. Since

$$\big(I-\hat{M}^{-1}\hat{S}(\gamma)\big)^{-1}-\gamma^{-1}I=-\gamma^{-1}\big(I-\hat{M}^{-1}\hat{S}(\gamma)\big)^{-1}\big\{\big(I-\hat{M}^{-1}\hat{S}(\gamma)\big)-\gamma I\big\}$$

and $\|(I-\hat{M}^{-1}\hat{S}(\gamma))^{-1}\|=O_p(1)$, the lemma is established. *Q.E.D.*

LEMMA SL.B.10: *Under the conditions of Theorem ST.B.2, as $n\to\infty$,*

$$\left\|\big(\hat{S}(\gamma)^{-1}X'^{*}(\gamma)\varepsilon-\gamma(1-\gamma)^{-1}\hat{M}^{-1}X'\varepsilon\big)-\hat{M}^{-1}\frac{\left(\sum_{t=1}^{[n\gamma]}\varepsilon_t x_t-\gamma\sum_{t=1}^{n}\varepsilon_t x_t\right)}{1-\gamma}\right\|$$

$$=O_p\big(\lambda_n^{-2}\sqrt{np}\varkappa_p\big). \tag{SB.49}$$

PROOF: First, note that

$$(1-\gamma)^{-1}\left(\sum_{t=1}^{[n\gamma]}\varepsilon_t x_t-\gamma\sum_{t=1}^{n}\varepsilon_t x_t\right)=(1-\gamma)^{-1}X'^{*}(\gamma)\varepsilon-\gamma(1-\gamma)^{-1}X'\varepsilon,$$

so the term inside the norm in (SB.49) equals

$$\begin{aligned}&\big(\hat{S}(\gamma)^{-1}-(1-\gamma)^{-1}\hat{M}^{-1}\big)X'^{*}(\gamma)\varepsilon\\&\quad=(1-\gamma)^{-1}\hat{M}^{-1}\big((1-\gamma)\hat{M}-\hat{S}(\gamma)\big)\hat{S}(\gamma)^{-1}X'^{*}(\gamma)\varepsilon.\end{aligned} \tag{SB.50}$$

The norm of the RHS of (SB.50) is bounded by a constant times

$$\big\|\hat{M}^{-1}\big\|\,\big\|\hat{S}(\gamma)^{-1}\big\|\left(\left\|n^{-1}\sum_{t=1}^{[n\gamma]}x_t x_t'-\gamma M\right\|+\|\hat{M}-M\|\right)\big\|X'^{*}(\gamma)\varepsilon\big\|=O_p\big(\lambda_n^{-2}\sqrt{np}\varkappa_p\big),$$

the last equality following from Assumptions 3, Lemma SL.B.2, and also (SB.25). *Q.E.D.*

APPENDIX S.C: PROOF OF THEOREM 4.2

PROOF: It is sufficient to check (4.7), whence (4.8) follows. Let $\varepsilon^{\star}$ denote the vector collecting $\varepsilon_t^{\star}=\hat{e}_t(\gamma)\xi_t$, where ξ_t is an iid sequence of Rademacher variables. Then

$$\hat{\delta}_2^{\star}(\gamma)=A(\gamma)\varepsilon^{\star},$$

since $\delta_2 = 0$ under $\mathcal{H}_0$. Also, we have

$$W_n^\star(\gamma) = n(\varepsilon^\star)' A'(\gamma)\hat{B}^\star(\gamma)^{-1}A(\gamma)\varepsilon^\star, \tag{SC.1}$$

where $\hat{B}^\star(\gamma) = R\hat{M}(\gamma)^{-1}\hat{\Omega}^\star(\gamma)\hat{M}(\gamma)^{-1}R'$ and $\hat{\Omega}^\star(\gamma)$ is constructed as $\hat{\Omega}(\gamma)$ with the bootstrap sample.

We begin with

$$\begin{aligned} E^\star \bar{W}_n^\star(\gamma) &= n \operatorname{tr} A'(\gamma)\hat{B}(\gamma)^{-1}A(\gamma)E^\star\varepsilon^\star(\varepsilon^\star)', \\ &= n \operatorname{tr} A'(\gamma)\hat{B}(\gamma)^{-1}A(\gamma)\operatorname{diag}[\hat{e}_1(\gamma)^2, \dots, \hat{e}_n(\gamma)^2], \end{aligned} \tag{SC.2}$$

where $\bar{W}_n^\star(\gamma) = n(\varepsilon^\star)'A'(\gamma)B(\gamma)^{-1}A(\gamma)\varepsilon^\star$. Note that the term in (SC.2) subtracted by p is $o_p(p^{1/2})$ uniformly in γ due to Lemma SL.B.7, Lemma SL.B.8, and Lemma SL.B.3.

Next, we show that the order of the difference between $E^\star\bar{W}_n^\star(\gamma)$ and $E^\star W_n^\star(\gamma)$ is $o_p(p^{1/2})$. Following (SB.39), write

$$E^\star\left|\bar{W}_n^\star(\gamma) - W_n^\star(\gamma)\right| \le E^\star\left(\left\|n^{-1/2}A'(\gamma)\varepsilon^\star\right\|^2\left\|\hat{B}(\gamma)^{-1} - \hat{B}^\star(\gamma)^{-1}\right\|\right).$$

To apply the Cauchy–Schwarz inequality, and to bound $E^\star\|\hat{B}(\gamma)^{-1} - \hat{B}^\star(\gamma)^{-1}\|^2$, we derive bounds for $E^\star\|\hat{B}^\star(\gamma)^{-1}\|^4$ and $E^\star\|\hat{B}(\gamma) - \hat{B}^\star(\gamma)\|^4$. Since both are similar to the derivations for the sample counterparts in Lemmas SL.B.1 and SL.B.5, we only illustrate the latter. Recall $\hat{B}(\gamma) - \hat{B}^\star(\gamma) = R'\hat{M}(\gamma)^{-1}(\hat{\Omega}(\gamma) - \hat{\Omega}^\star(\gamma))\hat{M}(\gamma)^{-1}R$ and $\sup_{\gamma\in\Gamma}\|\hat{M}(\gamma)^{-1}\| = O_p(\lambda_n^{-1})$ by Lemma SL.B.2. Following the steps in the proof of Lemma SL.B.1, the term $\hat{\Omega}(\gamma) - \hat{\Omega}^\star(\gamma)$ is given by the sum of $U_1^\star(\gamma)$ and $U_3^\star(\gamma)$ therein. Due to the triangle inequality and c_r inequality, we only show $E^\star\|U_j^\star(\gamma)\|^4 = O_p(\lambda_n^{-8}p^{12}/n^4)$, for $j = 1, 3$. Note that by the independence of the sequence ξ_t,

$$\begin{aligned} E^\star\left\|U_1^\star(\gamma)\right\|^4 &\le \left(n^{-1}\sum_{t=1}^n\left(x_t'(\gamma)x_t(\gamma)\right)^2\right)^4 E^\star\left(\left(\delta^\star - \hat{\delta}^\star(\gamma)\right)'\left(\delta^\star - \hat{\delta}^\star(\gamma)\right)\right)^4 \\ &\le O_p(p^8)\left\|\hat{M}(\gamma)^{-1}\right\|^8 n^{-8}\sum_{t_1,t_2,t_3,t_4}\hat{e}_{t_1}^2 x_{t_1}'x_{t_1}\cdots\hat{e}_{t_4}^2 x_{t_4}'x_{t_4}, \end{aligned}$$

to yield the desired result and the bound for $U_3^\star$ is similarly obtained. Putting these together yields $E^\star\|\hat{B}(\gamma)^{-1} - \hat{B}^\star(\gamma)^{-1}\|^2 = O_p(\lambda_n^{-10}p^{12}/n^4)$.

Next, similar to the preceding bound,

$$E^\star\left\|n^{-1/2}A'(\gamma)\varepsilon^\star\right\|^4 = O_p(\lambda_n^{-4})\left(n^{-1}\sum_{t=1}^n x_t'x_t\hat{e}_t^2\right)^2 = O_p(\lambda_n^{-4}p^2),$$

as ξ_t is an iid Rademacher sequence. Then, under the condition (3.1), $\lambda_n^{-14}p^{14}/n^4 = o(p)$ and this completes the proof. *Q.E.D.*

APPENDIX S.D: VERIFICATION OF COVARIANCE DECAY IN ASSUMPTION 4

TABLE S.TAB.D.1
$(n^4 p^2)^{-1} \sum_{t=1}^{n} \sum_{s=1}^{t-1} \mathrm{cov}(\mathrm{tr}(\Upsilon_t \Xi_t), \mathrm{tr}(\Upsilon_s \Xi_s))$ WITH $n = 999, \ldots, 9999$ FOR MULTIPLE REGRESSION.

type	α_x	α	999	3249	5499	7749	9999
1	0.1	0.3	0.0082	0.0058	0.0047	0.0043	0.0043
1	0.1	0.4	0.0086	0.0053	0.0044	0.0043	0.0038
1	0.1	0.5	0.0086	0.0055	0.0046	0.0041	0.0037
1	0.1	0.55	0.008	0.0054	0.0042	0.0037	0.0034
1	0.5	0.3	0.0377	0.0264	0.0233	0.0219	0.0212
1	0.5	0.4	0.0405	0.0251	0.0225	0.0197	0.0191
1	0.5	0.5	0.0357	0.0217	0.0174	0.0147	0.0138
1	0.5	0.55	0.038	0.0221	0.0171	0.0155	0.0138
1	0.7	0.3	0.2278	0.1735	0.1388	0.137	0.1354
1	0.7	0.4	0.2615	0.1858	0.1544	0.1413	0.1375
1	0.7	0.5	0.2757	0.1561	0.1416	0.1359	0.1313
1	0.7	0.55	0.2062	0.1484	0.1352	0.1149	0.1002
2	0.1	0.3	0.0155	0.0075	0.0054	0.0047	0.0044
2	0.1	0.4	0.011	0.0061	0.0054	0.0046	0.0042
2	0.1	0.5	0.0202	0.0108	0.0122	0.0069	0.006
2	0.1	0.55	0.0355	0.095	0.0199	0.0091	0.006
2	0.5	0.3	0.0518	0.0334	0.0265	0.0241	0.0219
2	0.5	0.4	0.0614	0.0307	0.0255	0.0241	0.0223
2	0.5	0.5	0.0703	0.0498	0.0302	0.0256	0.0238
2	0.5	0.55	1.6204	0.0992	0.0373	0.0313	0.0326
2	0.7	0.3	0.3164	0.203	0.188	0.1722	0.1536
2	0.7	0.4	0.3501	0.2266	0.2175	0.1972	0.183
2	0.7	0.5	0.9339	0.2196	0.8319	0.3447	0.3014
2	0.7	0.55	1.4368	0.5232	0.2518	0.1936	0.167
3	0.1	0.3	0.0078	0.0039	0.0035	0.0032	0.0028
3	0.1	0.4	0.0061	0.0035	0.0028	0.0024	0.0021
3	0.1	0.5	0.004	0.0023	0.0019	0.0018	0.0017
3	0.1	0.55	0.0035	0.0019	0.0014	0.0013	0.0012
3	0.5	0.3	0.0363	0.0232	0.0161	0.0141	0.0135
3	0.5	0.4	0.0229	0.0169	0.0134	0.0122	0.0108
3	0.5	0.5	0.0202	0.0122	0.0119	0.0099	0.0092
3	0.5	0.55	0.0172	0.01	0.0075	0.0067	0.006
3	0.7	0.3	0.2202	0.1438	0.1218	0.1062	0.0963
3	0.7	0.4	0.1665	0.107	0.0919	0.0849	0.0789
3	0.7	0.5	0.1273	0.0807	0.077	0.0696	0.062
3	0.7	0.55	0.1106	0.0556	0.0522	0.0453	0.0438

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