Supplement to "A Bayesian dynamic stochastic general equilibrium model of stock market bubbles and business cycles": Appendices

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This online material contains six appendices to the paper. Appendix A proves Proposition 1 in the paper. Appendix B derives the stationary equilibrium. Appendix C studies the bubbly steady state. Appendix D provides the log-linearized equilibrium system around the bubbly steady state. Appendix E presents a table of business cycle moments. Appendix F presents a robustness analysis.

Appendix A: Proof of Proposition 1 in the paper

We use a conjecture and verification strategy to find the decision rules at the firm level. We first study the optimal investment problem by fixing the capacity utilization rate u_t^j. Using (14) and (16), we can write firm j's dynamic programming problem as

$$\begin{aligned} v_t(\varepsilon_t^j)K_t^j + b_{t,\tau}(\varepsilon_t^j) - v_{Lt}(\varepsilon_t^j)L_t^j \\ = \max_{I_t^j, L_{t+1}^j} u_t^j R_t K_t^j - P_t I_t^j - L_t^j + \frac{L_{t+1}^j}{R_{ft}} \\ + Q_t\big[(1-\delta_t^j)K_t^j + \varepsilon_t^j I_t^j\big] + B_{t,\tau} - Q_{Lt} L_{t+1}^j \end{aligned} \tag{A.1}$$

subject to the investment constraint

$$0 \le P_t I_t^j \le u_t^j R_t K_t^j - L_t^j + \frac{L_{t+1}^j}{R_{ft}} + \eta_t K_t^j. \tag{A.2}$$

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For $\varepsilon_t^j \leq P_t/Q_t$, $I_t^j = 0$. Optimizing over L_{t+1}^j yields $Q_{Lt} = 1/R_{ft}$. For $\varepsilon_{t+1}^j \geq P_t/Q_t$, the optimal investment level must reach the upper bound in the above investment constraint. We can then immediately derive the optimal investment rule in (18). In addition, the credit constraint (17) must bind so that

$$\frac{1}{R_{ft}} L_{t+1}^j = Q_t \xi_t K_t^j + B_{t,\tau}. \tag{A.3}$$

Substituting the optimal investment rule and $Q_{Lt} = 1/R_{ft}$ into (A.1) yields

$$\begin{aligned} & v_t(\varepsilon_t^j) K_t^j + b_{t,\tau}(\varepsilon_t^j) - v_{Lt}(\varepsilon_t^j) L_t^j \\ & \quad = u_t^j R_t K_t^j + Q_t(1-\delta_t^j) K_t^j + B_{t,\tau} - L_t^j \\ & \qquad + \max\{Q_t \varepsilon_t^j / P_t - 1, 0\} \times \left(u_t^j R_t K_t^j + \eta_t K_t^j - L_t^j + \frac{L_{t+1}^j}{R_{ft}} \right). \end{aligned} \tag{A.4}$$

Since u_t^j is determined before observing ε_t^j, it solves the problem

$$\max_{u_t^j} u_t^j R_t K_t^j + Q_t(1-\delta_t^j) K_t^j + G_t u_t^j R_t K_t^j, \tag{A.5}$$

where G_t is defined by (20). We then obtain the first-order condition

$$R_t(1+G_t) = Q_t \delta'(u_t^j). \tag{A.6}$$

Since $\delta_t^j = \delta(u_t^j)$ is convex, this condition is also sufficient for optimality. From this condition, we can immediately deduce that optimal u_t^j does not depend on firm identity so that we can remove the superscript j.

By defining $\delta_t \equiv \delta(u_t)$, (A.4) becomes

$$\begin{aligned} & v_t(\varepsilon_t^j) K_t^j + b_{t,\tau}(\varepsilon_t^j) - v_{Lt}(\varepsilon_t^j) L_t^j \\ & \quad = u_t R_t K_t^j + Q_t(1-\delta_t) K_t^j + B_{t,\tau} - L_t^j \\ & \qquad + \max\{Q_t \varepsilon_t^j / P_t - 1, 0\} \times \left(u_t R_t K_t^j + \eta_t K_t^j - L_t^j + \frac{L_{t+1}^j}{R_{ft}} \right), \end{aligned}$$

where L_{t+1}^j / R_{ft} is given by (A.3). Matching coefficients yields

$$v_t(\varepsilon_t^j) = \begin{cases} u_t R_t + Q_t(1-\delta_t) & \\ \quad + (Q_t \varepsilon_t^j / P_t - 1)(u_t R_t + \eta_t + \xi_t Q_t), & \text{if } \varepsilon_t^j \geq \dfrac{P_t}{Q_t}, \\ u_t R_t + Q_t(1-\delta_t), & \text{otherwise}, \end{cases} \tag{A.7}$$

$$b_{t,\tau}(\varepsilon_t^j) = \begin{cases} (Q_t \varepsilon_t^j / P_t - 1) B_{t,\tau}, & \text{if } \varepsilon_t^j \geq \dfrac{P_t}{Q_t}, \\ B_{t,\tau}, & \text{otherwise}, \end{cases} \tag{A.8}$$

and

$$v_{Lt}(\varepsilon_t^j) = \begin{cases} Q_t \varepsilon_t^j / P_t - 1, & \text{if } \varepsilon_t^j \geq \dfrac{P_t}{Q_t}, \\ 1, & \text{otherwise.} \end{cases}$$

Using (14), we then obtain (21), and (22) and (23).

Appendix B: Stationary equilibrium

We define the transformed variables

$$\tilde{C}_t \equiv \frac{C_t}{\Gamma_t}, \qquad \tilde{I}_t \equiv \frac{I_t}{Z_t \Gamma_t}, \qquad \tilde{Y}_t \equiv \frac{Y_t}{\Gamma_t}, \qquad \tilde{K}_t \equiv \frac{K_t}{\Gamma_{t-1} Z_{t-1}},$$

$$\tilde{P}_t^s \equiv \frac{P_t^s}{\Gamma_t}, \qquad \tilde{B}_t^a \equiv \frac{B_t^a}{\Gamma_t}, \qquad \tilde{X}_t \equiv \frac{X_t}{\Gamma_t Z_t}, \qquad \tilde{W}_t \equiv \frac{W_t}{\Gamma_t},$$

$$\tilde{Q}_t \equiv Q_t Z_t, \qquad \tilde{P}_t = P_t Z_t, \qquad \tilde{R}_t = R_t Z_t, \qquad \tilde{\Lambda}_t \equiv \Lambda_t \Gamma_t,$$

where $\Gamma_t = Z_t^{\alpha/(1-\alpha)} A_t$. The other variables are stationary and there is no need to scale them. To be consistent with a balanced growth path, we also assume that $K_{0t} = \Gamma_{t-1} Z_{t-1} K_0$, where K_0 is a constant.

The six shocks in the model are given as follows.

1. The permanent TFP shock:

$$A_t^p = A_{t-1}^p \lambda_{at}, \qquad \ln \lambda_{at} = (1 - \rho_a) \ln \bar{\lambda}_a + \rho_a \ln \lambda_{a,t-1} + \varepsilon_{at}. \tag{B.1}$$

2. The transitory TFP shock:

$$\ln A_t^m = \rho_{a^m} \ln A_{t-1}^m + \varepsilon_{a^m,t}. \tag{B.2}$$

3. The IST shock:

$$Z_t = Z_{t-1} \lambda_{zt}, \qquad \ln \lambda_{zt} = (1 - \rho_z) \ln \bar{\lambda}_z + \rho_z \ln \lambda_{z,t-1} + \varepsilon_{zt}. \tag{B.3}$$

4. The sentiment shock:

$$\ln \theta_t = (1 - \rho_\theta) \bar{\theta} + \rho_\theta \ln \theta_{t-1} + \varepsilon_{\theta,t}. \tag{B.4}$$

5. The labor shock:

$$\ln \psi_t = (1 - \rho_\psi) \ln \bar{\psi} + \rho_\psi \ln \psi_{t-1} + \varepsilon_{\psi t}. \tag{B.5}$$

6. The financial shock:

$$\ln \zeta_t = (1 - \rho_\zeta) \ln \bar{\zeta} + \rho_\zeta \ln \zeta_{t-1} + \varepsilon_{\zeta t}.$$

Here, all innovations are mutually independent and are independently and identically distributed normal random variables.

Denote by $g_{\gamma t} \equiv \Gamma_t/\Gamma_{t-1}$ the growth rate of Γ_t. Denote by g_γ the nonstochastic steady state of $g_{\gamma t}$, satisfying

$$\ln g_\gamma \equiv \frac{\alpha}{1-\alpha} \ln \bar{\lambda}_z + \ln \bar{\lambda}_a. \tag{B.6}$$

On the nonstochastic balanced growth path, investment and capital grow at the rate of $\bar{\lambda}_I \equiv g_\gamma \bar{\lambda}_z$; consumption, output, wages, and bubbles grow at the rate of g_γ; and the rental rate of capital, Tobin's marginal Q, and the relative price of investment goods decrease at the rate $\bar{\lambda}_z$.

After the transformation described in Section 3, we can derive a system of 15 equations for 15 transformed variables: $\{\tilde{C}_t, \tilde{I}_t, \tilde{Y}_t, N_t, \tilde{K}_t, u_t, \tilde{Q}_t, \tilde{X}_t, \tilde{P}_t, \tilde{W}_t, \tilde{R}_t, m_t, \tilde{B}_t^a, R_{ft}, \tilde{\Lambda}_t\}$.

1. Resource constraint:

$$\tilde{C}_t + \left[1 + \frac{\Omega}{2}\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{zt} g_{\gamma t} - \bar{\lambda}_I\right)^2\right]\tilde{I}_t = \tilde{Y}_t, \tag{B.7}$$

where $g_{zt} = Z_t/Z_{t-1}$.

2. Aggregate investment:

$$\tilde{I}_t = \left(\alpha \tilde{Y}_t + \zeta_t \tilde{Q}_t \tilde{X}_t + \tilde{B}_t^a\right)\frac{1 - \Phi(\varepsilon_t^*)}{\tilde{P}_t}, \tag{B.8}$$

where $\varepsilon_t^* = \tilde{P}_t/\tilde{Q}_t$.

3. Aggregate output:

$$\tilde{Y}_t = (u_t \tilde{X}_t)^\alpha N_t^{1-\alpha}. \tag{B.9}$$

4. Labor supply:

$$(1-\alpha)\frac{\tilde{Y}_t}{N_t}\tilde{\Lambda}_t = \psi_t. \tag{B.10}$$

5. The law of motion for capital:

$$\tilde{K}_{t+1} = (1-\delta_t)\tilde{X}_t + \tilde{I}_t \frac{\Sigma(\varepsilon_t^*)}{1 - \Phi(\varepsilon_t^*)}, \tag{B.11}$$

where

$$\Sigma(\varepsilon_t^*) \equiv \int_{\varepsilon > \varepsilon_t^*} \varepsilon \, d\Phi(\varepsilon).$$

6. Capacity utilization:

$$\alpha \frac{\tilde{Y}_t}{u_t \tilde{X}_t}(1 + G_t) = \tilde{Q}_t \delta'(u_t), \tag{B.12}$$

where

$$G_t = \int_{\varepsilon > \varepsilon_t^*} \left(\varepsilon/\varepsilon_t^* - 1\right) d\Phi(\varepsilon) = \frac{\Sigma\left(\varepsilon_t^*\right)}{\varepsilon_t^*} + \Phi\left(\varepsilon_t^*\right) - 1.$$

7. Marginal Q:

$$\tilde{Q}_t = \beta(1-\delta_e)E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t} \frac{\tilde{Q}_{t+1}}{g_{zt+1}g_{\gamma t+1}} \left[u_{t+1}\delta'(u_{t+1}) + (1-\delta_{t+1}) + \zeta_{t+1}G_{t+1}\right]. \quad \text{(B.13)}$$

8. Effective capital stock used in production:

$$\tilde{X}_t = \frac{1-\delta_e}{g_{zt}g_{\gamma t}}\tilde{K}_t + \delta_e K_0. \quad \text{(B.14)}$$

9. Euler equation for investment goods producers:

$$\begin{aligned} \tilde{P}_t = {} & 1 + \frac{\Omega}{2}\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{zt}g_{\gamma t} - \bar{\lambda}_I\right)^2 + \Omega\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{zt}g_{\gamma t} - \bar{\lambda}_I\right)\frac{\tilde{I}_t}{\tilde{I}_{t-1}} g_{zt}g_{\gamma t} \\ & - \beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t}\Omega\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t} g_{zt+1}g_{\gamma t+1} - \bar{\lambda}_I\right)\left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t}\right)^2 g_{zt+1}g_{\gamma t+1}. \end{aligned} \quad \text{(B.15)}$$

10. The wage rate:

$$\tilde{W}_t = (1-\alpha)\frac{\tilde{Y}_t}{N_t}. \quad \text{(B.16)}$$

11. The rental rate of capital:

$$\tilde{R}_t = \frac{\alpha \tilde{Y}_t}{u_t \tilde{X}_t}. \quad \text{(B.17)}$$

12. Evolution of the number of bubbly firms:

$$m_t = m_{t-1}(1-\delta_e)\theta_{t-1} + \delta_e \omega. \quad \text{(B.18)}$$

13. Evolution of the total value of the bubble:

$$\tilde{B}_t^a = \beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t} \tilde{B}_{t+1}^a (1+G_{t+1})(1-\delta_e)\theta_t \frac{m_t}{m_{t+1}}. \quad \text{(B.19)}$$

14. The risk-free rate:

$$\frac{1}{R_{ft}} = \beta E_t \frac{\tilde{\Lambda}_{t+1}}{\tilde{\Lambda}_t} \frac{1}{g_{\gamma t+1}} (1+G_{t+1})(1-\delta_e). \quad \text{(B.20)}$$

15. Marginal utility for consumption:

$$\tilde{\Lambda}_t = \frac{1}{\tilde{C}_t - h\tilde{C}_{t-1}/g_{\gamma t}} - \beta E_t \frac{h}{\tilde{C}_{t+1}g_{\gamma t+1} - h\tilde{C}_t}. \quad \text{(B.21)}$$

Appendix C: Steady state

The transformed system presented in Appendix B has a nonstochastic steady state. We eliminate $\tilde{W}_t$ and $\tilde{R}_t$, and then obtain a system of 15 equations for 15 steady-state values $\{\tilde{C}, \tilde{I}, \tilde{Y}, N, \tilde{K}, u, \tilde{Q}, \tilde{X}, \tilde{P}, \tilde{W}, \tilde{R}, m, \tilde{B}^a, R_f, \tilde{\Lambda}\}$, where we have removed time subscripts. We assume that the function $\delta(\cdot)$ is such that the steady-state capacity utilization rate is equal to 1. In addition, we set $\tilde{Q} = 1$, which pins down G.

1. Resource constraint:

$$\tilde{C} + \tilde{I} = \tilde{Y}, \tag{C.1}$$

where we have used the fact that $\bar{\lambda}_I = \bar{\lambda}_z g_\gamma$.

2. Aggregate investment:

$$\tilde{I} = \left(\alpha \tilde{Y} + \bar{\zeta} \tilde{Q} \tilde{X} + \tilde{B}^a\right) \frac{1 - \Phi(\varepsilon^*)}{\tilde{P}}, \tag{C.2}$$

where $1 - \Phi(\varepsilon^*) = \int_{\varepsilon > \varepsilon^*} d\Phi(\varepsilon)$ and $\varepsilon^* = \tilde{P}/\tilde{Q}$.

3. Aggregate output:

$$\tilde{Y} = \tilde{X}^\alpha N^{1-\alpha}. \tag{C.3}$$

4. Labor supply:

$$(1 - \alpha) \frac{\tilde{Y}}{N} \tilde{\Lambda} = \bar{\psi}. \tag{C.4}$$

5. End-of-period capital stock:

$$\tilde{K} = \left(1 - \delta(1)\right) \tilde{X} + \tilde{I} \frac{\Sigma(\varepsilon^*)}{1 - \Phi(\varepsilon^*)}, \tag{C.5}$$

where

$$\Sigma(\varepsilon^*) \equiv \int_{\varepsilon > \varepsilon^*} \varepsilon \, d\Phi(\varepsilon).$$

6. Capacity utilization:

$$\alpha \frac{\tilde{Y}}{\tilde{X}} (1 + G) = \tilde{Q} \delta'(1), \tag{C.6}$$

where

$$G = \int_{\varepsilon > \varepsilon^*} \left(\varepsilon/\varepsilon^* - 1\right) d\Phi(\varepsilon) = \frac{\Sigma(\varepsilon^*)}{\varepsilon^*} + \Phi(\varepsilon^*) - 1.$$

7. Marginal Q:

$$1 = \beta(1 - \delta_e) \frac{1}{\bar{\lambda}_z g_\gamma} \left[\delta'(1) + 1 - \delta(1) + \bar{\zeta} G\right]. \tag{C.7}$$

8. Effective capital stock used in production:

$$\tilde{X} = \frac{1-\delta_e}{\bar{\lambda}_z g_\gamma}\tilde{K} + \delta_e K_0. \tag{C.8}$$

9. Euler equation for investment goods producers:

$$\tilde{P} = 1. \tag{C.9}$$

10. The wage rate:

$$\tilde{W} = (1-\alpha)\frac{\tilde{Y}}{N}. \tag{C.10}$$

11. The rental rate of capital:

$$\tilde{R} = \frac{\alpha\tilde{Y}}{\tilde{X}}. \tag{C.11}$$

12. Evolution of the number of bubbly firms:

$$m = m(1-\delta_e)\bar{\theta} + \delta_e\omega. \tag{C.12}$$

13. Evolution of the total value of the bubble:

$$\tilde{B}^a = \beta\tilde{B}^a(1+G)(1-\delta_e)\bar{\theta}. \tag{C.13}$$

14. The risk-free rate:

$$\frac{1}{R_f} = \beta\frac{1}{g_\gamma}(1+G)(1-\delta_e). \tag{C.14}$$

15. Marginal utility for consumption:

$$\tilde{\Lambda} = \frac{1}{\tilde{C} - h\tilde{C}/g_\gamma} - \frac{\beta h}{\tilde{C}g_\gamma - h\tilde{C}}. \tag{C.15}$$

For convenience, define $\varepsilon_t^* = P_t/Q_t = \tilde{P}_t/\tilde{Q}_t$ as the investment threshold. We use a variable without the time subscript to denote its steady-state value in the transformed stationary system. The following proposition characterizes the bubbly steady state.[1]

Proposition C1. *Suppose that $\omega > 0$ and $0 < \varepsilon_{\min} < \beta(1-\delta_e)\bar{\theta} < \beta$. Then there exists a unique steady-state threshold $\varepsilon^* \in (\varepsilon_{\min}, \varepsilon_{\max})$ satisfying*

$$\int_{\varepsilon>\varepsilon^*} \left(\varepsilon/\varepsilon^* - 1\right) d\Phi(\varepsilon) = \frac{1}{\beta(1-\delta_e)\bar{\theta}} - 1. \tag{C.16}$$

[1] The bubbleless steady state can be obtained by setting $\tilde{B}^a = 0$ and $m = \omega = 0$. In this case, we can remove (C.13) and (C.12).

If the parameter values are such that

$$\frac{\tilde{B}^a}{\tilde{Y}} = \frac{\left[\varphi_k - \left(1-\delta(1)\right)\right]\varphi_x}{1/\left[\beta(1-\delta_e)\bar{\theta}\right] - \Phi\left(\varepsilon^*\right)} - \alpha - \bar{\zeta}\varphi_x > 0, \tag{C.17}$$

where we define

$$\varphi_k \equiv \left(\frac{1-\delta_e}{\bar{\lambda}_z g_\gamma} + \delta_e \frac{K_0}{\tilde{K}}\right)^{-1}, \tag{C.18}$$

$$\varphi_x \equiv \frac{\alpha}{\bar{\lambda}_z g_\gamma \bar{\theta} - \left(1-\delta(1)\right)\beta(1-\delta_e)\bar{\theta} - \bar{\zeta}\left[1-\beta(1-\delta_e)\bar{\theta}\right]}, \tag{C.19}$$

then there exists a unique bubbly steady-state equilibrium with the bubble–output ratio given in (C.17). *The steady-state growth rate of the bubble is given by* $\bar{\theta} = R_f/g_\gamma$, *where* R_f *is the steady-state interest rate. In addition, if*

$$\delta'(1) = \frac{\alpha}{\beta(1-\delta_e)\bar{\theta}} \frac{1}{\varphi_x}, \tag{C.20}$$

then the capacity utilization rate in this steady state is equal to 1.

Proof. In the steady state, (B.15) implies that $\tilde{P} = 1$. Hence, by definition, we have $\varepsilon^* = 1/\tilde{Q}$. Then by the evolution equation (B.19) of the total bubble, we obtain the steady-state relation

$$\frac{1}{\beta(1-\delta_e)\bar{\theta}} - 1 = G = \int_{\varepsilon>\varepsilon^*} \left(\varepsilon/\varepsilon^* - 1\right) d\Phi(\varepsilon). \tag{C.21}$$

Define the expression on the right-hand side of the last equality as a function of ε^*: $G(\varepsilon^*)$. Then we have $G(\varepsilon_{\min}) = \frac{1}{\varepsilon_{\min}} - 1$ and $G(\varepsilon_{\max}) = 0$. Given the assumption that $\varepsilon_{\min} < \beta(1-\delta_e)\bar{\theta}$, there is a unique solution ε^* to (C.21) by the intermediate value theorem. In addition, by the definition of G, we have

$$G = \frac{\Sigma\left(\varepsilon^*\right)}{\varepsilon^*} - \left[1-\Phi\left(\varepsilon^*\right)\right],$$

where $\Sigma(\varepsilon^*) = \int_{\varepsilon>\varepsilon^*} \varepsilon \, d\Phi(\varepsilon)$. Thus, $\Sigma(\varepsilon^*)$ can be expressed as

$$\Sigma\left(\varepsilon^*\right) = \left[G + 1 - \Phi\left(\varepsilon^*\right)\right]\varepsilon^*. \tag{C.22}$$

Suppose that the steady-state capacity utilization rate is equal to 1. The steady-state version of (B.13) gives (C.7) and the steady-state version of (B.12) gives (C.6). Using these two equations, we can derive

$$\alpha\frac{\tilde{Y}}{\tilde{X}} = \frac{\tilde{Q}}{1+G}\left[\frac{g_z g_\gamma}{\beta(1-\delta_e)} - \left(1-\delta(1)\right) - \bar{\zeta} G\right]. \tag{C.23}$$

Substituting (C.21) into the above equation yields

$$\frac{\tilde{Q}\tilde{X}}{\tilde{Y}} = \varphi_x, \tag{C.24}$$

where φ_x is given by (C.19). To support the steady state $u=1$, we use (B.12) and (C.24) to show that condition (C.20) must be satisfied.

From (B.14), the end-of-period capital stock to the output ratio in the steady state satisfies

$$\frac{\tilde{K}}{\tilde{Y}}=\varphi_k\frac{\tilde{X}}{\tilde{Y}}, \tag{C.25}$$

where φ_k is given by (C.18). Then from (B.11), we can derive the steady-state relation

$$\begin{aligned}\frac{\tilde{I}}{\tilde{Y}}&=\frac{1-\Phi(\varepsilon^*)}{\Sigma(\varepsilon^*)}\big[\varphi_k-\big(1-\delta(1)\big)\big]\frac{\tilde{X}}{\tilde{Y}}\\&=\frac{1-\Phi(\varepsilon^*)}{\big[G+1-\Phi(\varepsilon^*)\big]}\big[\varphi_k-\big(1-\delta(1)\big)\big]\frac{\tilde{Q}\tilde{X}}{\tilde{Y}}\\&=\frac{\big[1-\Phi(\varepsilon^*)\big]\big[\varphi_k-\big(1-\delta(1)\big)\big]\varphi_x}{G+1-\Phi(\varepsilon^*)},\end{aligned} \tag{C.26}$$

where the second line follows from (C.22) and $\varepsilon^*=1/\tilde{Q}$, and the last line follows from (C.24). After substituting (C.21) into the above equation, we solve for $1-\Phi(\varepsilon^*)$:

$$1-\Phi(\varepsilon^*)=\frac{1/\big[\beta(1-\delta_e)\bar{\theta}\big]-1}{(\tilde{I}/\tilde{Y})^{-1}\big[\varphi_k-\big(1-\delta(1)\big)\big]\varphi_x-1}. \tag{C.27}$$

From (B.8), the steady-state total value of bubble to GDP ratio is given by

$$\frac{\tilde{B}^a}{\tilde{Y}}=\frac{\tilde{I}}{\tilde{Y}}\frac{1}{1-\Phi(\varepsilon^*)}-\alpha-\bar{\zeta}\frac{\tilde{Q}\tilde{X}}{\tilde{Y}}.$$

Substituting (C.21), (C.26), and (C.24) into the above equation yields (C.17). We require $\tilde{B}^a/\tilde{Y}>0$. By (23) and (34), the growth rate of bubbles of the surviving firms in the steady state is given by $\bar{\theta}=R_f/g_\gamma$. □

Appendix D: Log-linearized system

We eliminate equations for $\tilde{W}_t$ and $\tilde{R}_t$. The log-linearized system for 13 variables $\{\tilde{C}_t,\tilde{I}_t,\tilde{Y}_t,N_t,\tilde{K}_t,u_t,\tilde{Q}_t,\tilde{X}_t,\tilde{P}_t,m_t,\tilde{B}^a_t,R_{ft},\tilde{\Lambda}_t\}$, including two growth rates, are summarized as follows.

1. Resource constraint:

$$\hat{Y}_t=\frac{\tilde{C}}{\tilde{Y}}\hat{C}_t+\frac{\tilde{I}}{\tilde{Y}}\hat{I}_t. \tag{D.1}$$

2. Aggregate investment:

$$\begin{aligned}\hat{I}_t&=\frac{\alpha}{\alpha+\bar{\zeta}\varphi_x+\tilde{B}^a/\tilde{Y}}\hat{Y}_t+\frac{\bar{\zeta}\varphi_x}{\alpha+\bar{\zeta}\varphi_x+\tilde{B}^a/Y}(\hat{\zeta}_t+\hat{Q}_t+\hat{X}_t)\\&\quad+\frac{\tilde{B}^a/\tilde{Y}}{\alpha+\bar{\zeta}\varphi_x+\tilde{B}^a/\tilde{Y}}\hat{B}^a_t-\mu\hat{\varepsilon}^*_t-\hat{P}_t,\end{aligned} \tag{D.2}$$

where

$$\mu = \frac{\phi(\varepsilon^*)\varepsilon^*}{1 - \Phi(\varepsilon^*)}, \qquad \hat{\varepsilon}_t^* = \hat{P}_t - \hat{Q}_t. \tag{D.3}$$

3. Aggregate output:

$$\tilde{Y}_t = \alpha(\hat{u}_t + \hat{X}_t) + (1-\alpha)\hat{N}_t. \tag{D.4}$$

4. Labor supply:

$$\hat{\Lambda}_t + \hat{Y}_t - \hat{N}_t = \hat{\psi}_t. \tag{D.5}$$

5. End of period the capital stock:

$$\hat{K}_{t+1} = -\frac{\delta'(1)}{\varphi_k}\hat{u}_t + \frac{1-\delta(1)}{\varphi_k}\hat{X}_t + \left(1 - \frac{1-\delta(1)}{\varphi_k}\right)\left(\hat{I}_t - \frac{\mu}{\varphi_G}\hat{\varepsilon}_t^*\right), \tag{D.6}$$

where

$$\varphi_G \equiv -\frac{1-\Phi(\varepsilon^*)}{G} - 1. \tag{D.7}$$

6. Capacity utilization:

$$\hat{Y}_t - \hat{X}_t + \left[1 - \beta(1-\delta_e)\bar{\theta}\right]\varphi_G\hat{\varepsilon}_t^* = \hat{Q}_t + \left(1 + \frac{\delta''(1)}{\delta'(1)}\right)\hat{u}_t. \tag{D.8}$$

7. Marginal Q:

$$\begin{aligned}\hat{Q}_t = {} & E_t(\hat{\Lambda}_{t+1} - \hat{\Lambda}_t) + E_t(\hat{Q}_{t+1} - \hat{g}_{zt+1} - \hat{g}_{\gamma t+1}) \\ & + \frac{\beta(1-\delta_e)\delta'(1)}{\bar{\lambda}_z g_\gamma}\frac{\delta''(1)}{\delta'(1)}E_t\hat{u}_{t+1} \\ & + \frac{\bar{\zeta}\beta(1-\delta_e)G}{\bar{\lambda}_z g_\gamma}E_t\left(\hat{\zeta}_{t+1} + \varphi_G\hat{\varepsilon}_{t+1}^*\right).\end{aligned} \tag{D.9}$$

8. Effective capital stock:

$$\hat{X}_t = \frac{1-\delta_e}{\bar{\lambda}_z g_\gamma}\varphi_k(\hat{K}_t - \hat{g}_{zt} - \hat{g}_{\gamma t}). \tag{D.10}$$

9. Euler equation for investment goods producers:

$$\begin{aligned}\hat{P}_t = E_t\big[& (1+\beta)\Omega g_\gamma^2\bar{\lambda}_z^2\hat{I}_t + \Omega\bar{\lambda}_z^2 g_\gamma^2(\hat{g}_{\gamma t} + \hat{g}_{zt}) \\ & - \Omega\bar{\lambda}_z^2 g_\gamma^2\hat{I}_{t-1} - \beta\Omega\bar{\lambda}_z^2 g_\gamma^2(\hat{I}_{t+1} + \hat{g}_{zt+1} + \hat{g}_{\gamma t+1})\big].\end{aligned} \tag{D.11}$$

10. Evolution of the number of bubbly firms:

$$\hat{m}_t = (1-\delta_e)\bar{\theta}\hat{m}_{t-1} + (1-\delta_e)\bar{\theta}\hat{\theta}_{t-1}. \tag{D.12}$$

11. Evolution of the total value of the bubble:

$$\hat{B}_t^a = E_t\big(\hat{\Lambda}_{t+1} - \hat{\Lambda}_t + \hat{B}_{t+1}^a\big) + \big[1 - \beta(1-\delta_e)\bar{\theta}\big]\varphi_G E_t \hat{\varepsilon}_{t+1}^* + \frac{1-(1-\delta_e)\bar{\theta}}{(1-\delta_e)\bar{\theta}} E_t \hat{m}_{t+1}. \tag{D.13}$$

12. The risk-free rate:

$$-\hat{R}_{ft} = E_t(\hat{\Lambda}_{t+1} - \hat{\Lambda}_t - \hat{g}_{\gamma t+1}) + \big[1 - \beta(1-\delta_e)R_f/g_\gamma\big]\varphi_G E_t \hat{\varepsilon}_{t+1}^*. \tag{D.14}$$

13. Marginal utility for consumption:

$$\hat{\Lambda}_t = \frac{g_\gamma}{g_\gamma - \beta h}\bigg[-\frac{g_\gamma}{g_\gamma - h}\hat{C}_t + \frac{h}{g_\gamma - h}(\hat{C}_{t-1} - \hat{g}_{\gamma t})\bigg] - \frac{\beta h}{g_\gamma - \beta h} E_t\bigg[-\frac{g_\gamma}{g_\gamma - h}(\hat{C}_{t+1} + \hat{g}_{\gamma t+1}) + \frac{h}{g_\gamma - h}\hat{C}_t\bigg]. \tag{D.15}$$

14. The growth rate of consumption goods:

$$\hat{g}_{\gamma t} = \frac{\alpha}{1-\alpha}\hat{\lambda}_{zt} + \big(\hat{\lambda}_{at} + \hat{A}_t^m - \hat{A}_{t-1}^m\big). \tag{D.16}$$

15. The growth rate of the investment goods price:

$$\hat{g}_{zt} = \hat{\lambda}_{zt}. \tag{D.17}$$

In the above system, G is determined by (C.13),

$$G = \frac{1}{\beta(1-\delta_e)\bar{\theta}} - 1, \tag{D.18}$$

$(1 - \Phi(\varepsilon^*))$ is given by (C.27), and $\delta'(1)$ satisfies (C.20). The log-linearized shock processes are listed below.

1. The permanent technology shock:

$$\hat{\lambda}_{at} = \rho_a \hat{\lambda}_{at-1} + \varepsilon_{at}. \tag{D.19}$$

2. The transitory technology shock:

$$\hat{A}_t^m = \rho_{a^m} \hat{A}_{t-1}^m + \varepsilon_{a^m,t}. \tag{D.20}$$

3. The permanent investment-specific technology shock:

$$\hat{\lambda}_{zt} = \rho_z \hat{\lambda}_{zt-1} + \varepsilon_{zt}. \tag{D.21}$$

4. The labor supply shock:

$$\hat{\psi}_t = \rho_\psi \hat{\psi}_{t-1} + \varepsilon_{\psi t}. \tag{D.22}$$

5. The financial shock:

$$\hat{\zeta}_t = \rho_\zeta \hat{\zeta}_{t-1} + \varepsilon_{\zeta t}. \tag{D.23}$$

6. The sentiment shock:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \varepsilon_{\theta t}. \tag{D.24}$$

Appendix E: Business cycle moments

To evaluate our model performance, we present in Table S1 the baseline model's predictions regarding standard deviations, correlations with output, and serial correlations of output, consumption, investment, hours, and stock prices. This table also presents results for four estimated comparison models discussed in the paper. The model moments are computed using the simulated data from the estimated model when all shocks are turned on. We take the posterior modes as parameter values. Both simulated and actual data are in logs and are HP-filtered.

Appendix F: Robustness

F.1 *Extended model with consumer sentiment index*

Table S2 reports the prior and posterior distributions of estimated parameters in the extended model of Section 5, with the consumer sentiment index as one of the observation series. The parameters $\{a_j, b_j\}_{j=1}^5$ in the table are coefficients in the equation for the sentiment shock and in the observation equation of the consumer sentiment index. The variable σ_{err} represents the standard deviation of the measurement error.

F.2 *Priors*

In our baseline model of the paper, we choose 10 percent as the prior mean of σ_θ because we know that the stock market volatility is very high. To see if our result is robust to a smaller prior mean of σ_θ, we set the prior as Inv-Gamma with mean 0.01 and standard deviation infinite. We redo Bayesian estimation and report estimation results in Table S3. We find that these results are very similar to those in the baseline estimation.

F.3 *A hybrid model*

Our baseline model has abstracted away from many other potentially important shocks such as news shocks or uncertainty shocks. Thus, it is possible that the sentiment shock is not important at all in explaining stock prices and real variables if other shocks are taken into account. To examine this possibility, we follow the methodology of Ireland (2004) and combine the DSGE model with the VAR model. We then

TABLE S1. Business cycles statistics.

	Y	C	I	N	SP	P
			Standard Deviations (%)			
U.S. data	1.70	0.93	4.19	1.79	10.82	1.11
Baseline model	1.84	1.46	4.29	1.30	10.58	1.06
No stock price	1.15	0.95	3.04	1.11	1.32	1.22
No sentiment	1.50	1.40	3.32	1.57	10.20	2.49
No bubble	1.77	1.65	4.10	1.87	10.28	2.61
Extended	2.46	1.94	5.35	1.25	12.20	1.12
			Standard Deviations Relative to Y			
U.S. data	1.00	0.55	2.47	1.05	6.36	0.65
Baseline model	1.00	0.79	2.32	0.70	5.74	0.58
No stock price	1.00	0.83	2.63	0.96	1.15	1.06
No sentiment	1.00	0.93	2.21	1.04	6.78	1.65
No bubble	1.00	0.93	2.32	1.06	5.81	1.48
Extended	1.00	0.79	2.17	0.51	4.96	0.45
			First-Order Autocorrelations			
U.S. data	0.90	0.90	0.87	0.93	0.77	0.86
Baseline model	0.89	0.93	0.79	0.78	0.76	0.85
No stock price	0.83	0.89	0.73	0.77	0.72	0.88
No sentiment	0.91	0.91	0.83	0.74	0.72	0.81
No bubble	0.94	0.94	0.87	0.78	0.72	0.75
Extended	0.91	0.94	0.84	0.84	0.76	0.86
			Correlation With Y			
U.S. data	1.00	0.93	0.97	0.82	0.42	−0.13
Baseline model	1.00	0.94	0.88	0.61	0.39	−0.07
No stock price	1.00	0.88	0.80	0.68	0.45	−0.08
No sentiment	1.00	0.85	0.74	0.56	0.06	−0.14
No bubble	1.00	0.90	0.71	0.52	0.08	0.07
Extended	1.00	0.96	0.91	0.64	0.50	−0.08

Note: The model moments are computed using the simulated data (20,000 periods) from the estimated model at the posterior mode. All series are logged and detrended with the HP filter. The columns labeled Y, C, I, N, SP, and P refer, respectively, to output, consumption, investment, hours worked, the stock price, and the relative price of investment goods. “No bubble” corresponds to the model without bubbles. “No sentiment” corresponds to the baseline model without sentiment shocks. “No stock price” corresponds to the baseline model without using the stock price data in estimation. “Extended” corresponds to the model in Section 5.

estimate this hybrid model using Bayesian methods.[2] Following Ireland (2004), we now shut down all the shocks in the baseline model except the sentiment shock and introduce four measurement errors into the measurement equations for the data $\{\Delta P_t^{s\text{Data}}, \Delta C_t^{\text{Data}}, \Delta I_t^{\text{Data}}, \ln N^{\text{Data}}\}$. Specifically, let

$$\begin{bmatrix} \Delta P_t^{s\text{Data}} \\ \Delta C_t^{\text{Data}} \\ \Delta I_t^{\text{Data}} \\ \ln N^{\text{Data}} \end{bmatrix} = \begin{bmatrix} \Delta \hat{P}_t^s \\ \Delta \hat{C}_t \\ \Delta \hat{I}_t \\ \hat{N}_t \end{bmatrix} + \begin{bmatrix} \ln(g_\gamma) \\ \ln(g_\gamma) \\ \ln(g_\gamma) \\ \ln(\bar{N}) \end{bmatrix} + \boldsymbol{\nu}_t, \tag{F.1}$$

[2]We thank Tao Zha for suggesting that we conduct this analysis.

TABLE S2. Priors and posteriors of estimated parameters in the extended model.

	Prior Distribution			Posterior Distribution			
Parameter	Distr.	Mean	Std. Dev.	Mode	Mean	5%	95%
h	Beta	0.33	0.24	0.57	0.56	0.50	0.63
Ω	Gamma	2	2	0.03	0.04	0.01	0.06
δ''/δ'	Gamma	1	1	15.66	15.57	11.64	19.43
$\bar{\zeta}$	Beta	0.3	0.1	0.25	0.25	0.21	0.29
μ	Gamma	2	2	2.64	2.74	2.34	3.15
f_1	Gamma	1	1	0.08	0.07	0.01	0.13
f_2	Gamma	1	1	5.26	6.13	3.72	8.50
f_3	Gamma	1	1	0.67	0.62	0.00	1.10
a_1	Gamma	10	3	6.16	6.68	3.45	9.72
a_2	Gamma	10	3	13.48	14.75	9.06	19.96
a_3	Gamma	10	3	8.20	8.84	4.99	12.2
a_4	Gamma	10	3	5.36	5.58	3.29	7.97
a_5	Gamma	10	3	3.89	3.98	2.23	5.56
b_1	Gamma	2	2	0.25	0.25	0.19	0.30
b_2	Gamma	2	2	2.91	2.65	0.62	4.52
b_3	Gamma	2	2	3.79	3.63	1.69	5.36
b_4	Gamma	2	2	1.66	2.19	0.58	3.81
b_5	Gamma	2	2	0.38	0.93	0.08	1.83
ρ_a	Beta	0.5	0.2	0.68	0.67	0.52	0.80
ρ_{a^m}	Beta	0.5	0.2	0.81	0.80	0.73	0.87
ρ_z	Beta	0.5	0.2	0.40	0.38	0.26	0.51
ρ_θ	Beta	0.5	0.2	0.96	0.95	0.94	0.96
ρ_ψ	Beta	0.5	0.2	0.96	0.96	0.95	0.97
ρ_ζ	Beta	0.5	0.2	0.97	0.96	0.95	0.98
σ_a (%)	Inv-Gamma	1	Inf	0.74	0.75	0.59	0.91
σ_{a^m} (%)	Inv-Gamma	1	Inf	0.66	0.67	0.56	0.77
σ_z (%)	Inv-Gamma	1	Inf	0.59	0.60	0.53	0.67
σ_θ (%)	Inv-Gamma	1	Inf	13.70	13.80	10.28	17.05
σ_ψ (%)	Inv-Gamma	1	Inf	0.80	0.81	0.71	0.90
σ_ζ (%)	Inv-Gamma	1	Inf	0.76	0.70	0.39	0.97
σ_{err} (%)	Inv-Gamma	1	Inf	8.63	8.76	7.87	9.71

where $\boldsymbol{\nu}_t$ is the vector that contains four measurement errors, g_γ is the gross growth rate of output, and $\bar{N}$ is the average hours in the data. Following Ireland (2004), we assume that the measurement errors $\boldsymbol{\nu}_t$ follow a VAR(1) process

$$\boldsymbol{\nu}_t = \mathcal{A}\boldsymbol{\nu}_{t-1} + \mathcal{B}\hat{\varepsilon}_{\nu,t}, \tag{F.2}$$

where $\mathcal{A}$ is the coefficient matrix and $\mathcal{B}$ is assumed to be lower triangular such that the innovations in $\hat{\varepsilon}_{\nu,t}$ are orthogonal to each other.

The measurement errors in (F.2) can be considered as a combination of all omitted structural shocks in our baseline model and allow for potential model misspecifi-

TABLE S3. Prior and posterior distributions of parameters.

Parameter	Prior Distribution			Posterior Distribution			
	Distr.	Mean	Std. Dev.	Mode	Mean	5%	95%
h	Beta	0.33	0.24	0.54	0.54	0.49	0.61
Ω	Gamma	2	2	0.03	0.03	0.01	0.06
δ''/δ'	Gamma	1	1	11.44	11.92	8.33	15.49
$\bar{\zeta}$	Beta	0.3	0.1	0.29	0.30	0.22	0.36
μ	Gamma	2	2	2.57	2.60	2.12	3.19
f_1	Gamma	1	1	0.05	0.04	0.01	0.07
f_2	Gamma	1	1	4.73	4.82	2.54	7.08
f_3	Gamma	1	1	0.42	0.32	0.00	0.56
ρ_a	Beta	0.5	0.2	0.96	0.97	0.94	0.99
ρ_{a^m}	Beta	0.5	0.2	0.97	0.96	0.95	0.98
ρ_z	Beta	0.5	0.2	0.36	0.34	0.22	0.46
ρ_θ	Beta	0.5	0.2	0.93	0.92	0.90	0.95
ρ_ψ	Beta	0.5	0.2	0.99	0.98	0.96	0.99
ρ_ζ	Beta	0.5	0.2	0.88	0.87	0.81	0.94
σ_a (%)	Inv-Gamma	0.01	Inf	0.23	0.23	0.18	0.29
σ_{a^m} (%)	Inv-Gamma	0.01	Inf	1.03	1.04	0.93	1.16
σ_z (%)	Inv-Gamma	0.01	Inf	0.59	0.60	0.54	0.67
σ_θ (%)	Inv-Gamma	0.01	Inf	*17.85*	*19.46*	*11.65*	*26.38*
σ_ψ (%)	Inv-Gamma	0.01	Inf	0.81	0.82	0.72	0.93
σ_ζ (%)	Inv-Gamma	0.01	Inf	0.77	0.84	0.42	1.21

cations. We allow the measurement errors to be flexible enough so that the data are not necessarily driven by the sentiment shock. The idea is that if the sentiment shock is not the driving force, then (F.1) and (F.2) form a first-order Bayesian VAR system and the measurement errors should be important in explaining fluctuations in the data of $\{\Delta P_t^{s\text{Data}}, \Delta C_t^{\text{Data}}, \Delta I_t^{\text{Data}}, \ln N^{\text{Data}}\}$. On the other hand, if the baseline model is correctly specified and the sentiment shock is the main source of fluctuations, then the estimated measurement errors will be unimportant.

The variance decomposition shows that the sentiment shock remains the single most important factor accounting for the stock price variation, although its importance is somewhat reduced. It explains about 82 percent of the variation in the stock prices. It still accounts for significant fractions of fluctuations in investment, consumption, and output, explaining about 26, 38, and 35 percent, respectively. As in the baseline model, the sentiment shock is not important in explaining the fluctuation in hours. We also find that the estimates of the common parameters in the hybrid model are very similar to those in the baseline model. The smoothed sentiment shock is still highly correlated with the consumer sentiment data; the correlation is about 0.73. These results suggest that the importance of the sentiment shock is robust to the model variation and specification of different shocks.

Additional reference

Ireland, P. N. (2004), "A method for taking models to the data." *Journal of Economic Dynamics and Control*, 28, 1205–1226. [12, 13, 14]

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